

Exploring de Sitter Space and Holography

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Abstract

We explore aspects of the physics of de Sitter (dS) space that are relevant to holography with a positive cosmological constant. First we display a nonlocal map that commutes with the de Sitter isometries, transforms the bulk-boundary propagator and solutions of free wave equations in de Sitter onto the same quantities in Euclidean anti-de Sitter (EAdS), and takes the two boundaries of dS to the single EAdS boundary via an antipodal identification. Second we compute the action of scalar fields on dS as a functional of boundary data. Third, we display a family of solutions to 3d gravity with a positive cosmological constant in which the equal time sections are arbitrary genus Riemann surfaces, and compute the action of these spaces as a functional of boundary data from the Einstein gravity and Chern-Simons gravity points of view. These studies suggest that if de Sitter space is dual to a Euclidean conformal field theory (CFT), this theory should involve two disjoint, but possibly entangled factors. We argue that these CFTs would be of a novel form, with unusual hermiticity conditions relating left movers and right movers. After exploring these conditions in a toy model, we combine our observations to propose that a holographic dual description of de Sitter space would involve a pure entangled state in a product of two of our unconventional CFTs associated with the de Sitter boundaries. This state can be constructed to preserve the de Sitter symmetries and its decomposition in a basis appropriate to antipodal inertial observers would lead to the thermal properties of static patch.

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1 Introduction

Recent work has studied the possibility that de Sitter (dS) space is holographically dual to a Euclidean field theory defined on the late and early time conformal boundaries (\mathcal{I}^\pm) of this universe (see, e.g., [1, 2, 3, 4, 5, 6, 7]). While the status of the duality remains unclear, exploration of this possibility has led to many interesting new results. Amongst other successes, we have learned much about the asymptotic symmetries of de Sitter space (e.g., see [4, 5, 6]), we now have a method of measuring the gravitational mass of asymptotically de Sitter universes (e.g., [5, 6, 8, 9]), and we have understood new aspects of the famous ambiguity in choosing in the vacuum state of fields in de Sitter space [7]. These developments have been aided by a simple fact: de Sitter space ($\Lambda > 0$) and Euclidean AdS space (EAdS, $\Lambda < 0$) are both hyperboloids

$$-x_0^2 + x_1^2 + \cdots + x_d^2 = l^2 \quad ; \quad |\Lambda| = \frac{(d-1)(d-2)}{(2l^2)} \quad (1)$$

embedded in flat space with signature (1, d). As a result, analytic continuation of various facts about classical AdS leads to facts about classical dS – for example, we can learn that the asymptotic symmetry group of dS like AdS is the conformal group following [10, 11, 12, 13, 14, 15, 4]. However, the quantum mechanical physics of de Sitter space does not follow from analytic continuation – for example, the Green function for a scalar field obtained by analytic continuation from Euclidean AdS is the not the two point function of the scalar field in a de Sitter vacuum state [7].

Many interesting puzzles remain. Most importantly we do not yet have a controlled de Sitter background in string theory, or a well-understood soliton whose near horizon limit leads to a definition of de Sitter holography. Indeed, it is not clear that a stable de Sitter vacuum solution can be achieved.¹

Even given a de Sitter background, the fact that the boundaries of de Sitter space are euclidean surfaces at the beginning and end of time completely changes the structure of potential holographic bulk/boundary relationships. Application of intuitions arising from the AdS/CFT correspondence to de Sitter space also leads to questions about the unitarity of possible dual field theories.

¹The papers of Hull [1], and Hull and Khuri [16] find de Sitter backgrounds in a variety of string theories with unconventional signatures that are obtained via T-duality on timelike directions. While the stability of these theories remains in question, they present an interesting potential direction. Another unconventional, but potentially unstable approach appears in [17]. It has been pointed out that it might be possible to obtain de Sitter space from a non-critical string theory [18, 19]. One approach to Euclidean de Sitter space was suggested in [2]. Various discussions of de Sitter space in the context of supergravity can be found in [20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 29]. Related developments in the the study of time dependent backgrounds of string theory include recent work on spacelike brane solutions [31], space-time orbifolds with fixed planes that are localized in time [32], and Sen’s proposal that dynamical rolling of the tachyon of open string field theory can lead to interesting cosmologies [33].

Given these puzzles, we explore aspects of de Sitter physics that are relevant to holography with a positive cosmological constant, and uncover several new results. In Sec. 2 we recall how the AdS/CFT correspondence is formulated and explain why a naive translation to de Sitter will not work. We then display a non-local map that commutes with the de Sitter isometries, transforms the bulk-boundary propagator and solutions of free wave equations in de Sitter onto the same quantities in Euclidean AdS, and takes the two boundaries of dS to the single EAdS boundary via an antipodal identification reminiscent of the one advocated in [4]. The existence of this map might suggest that at the free field level some aspects of de Sitter physics could be captured by the Euclidean field theory dual to Euclidean AdS. However, the map has a non-trivial kernel, suggesting that not all of de Sitter physics can be captured in this way.

Keeping the AdS/CFT correspondence in mind, we proceed to study the action for scalar fields in de Sitter space as a functional of boundary data. To extend this investigation to gravity, we display a family of solutions to 3d gravity with a positive cosmological constant in which equal time sections are arbitrary genus Riemann surfaces, and compute the action for these spaces as a functional of boundary data in both the Einstein and Chern-Simons formulations of 3d gravity. In all these cases the boundary functional becomes trivial unless the contributions of data at each of the de Sitter boundaries is treated independently.

These studies, along with the non-local map discussed above, suggest that if de Sitter space is dual to a Euclidean CFT, this theory should involve two disjoint, but possibly entangled factors. We argue that these CFTs would be of a novel form, with unusual hermiticity conditions relating left movers and right movers. In Sec. 3 we explore how such hermiticity conditions are implemented in a simple model with two chiral bosons. We combine our observations to propose that a holographic dual description of de Sitter space would involve a pure entangled state in a product of two of our unconventional CFTs associated with the de Sitter boundaries. In particular, we show that it is possible to construct such a state which is invariant under a single $SL(2, C)$ in a product of two theories that are separately $SL(2, C)$ invariant, as would be necessary for a holographic description of 3d de Sitter. The decomposition of this state in a basis appropriate to antipodal inertial observers would lead to the thermal properties of the static patch. In Sec. 4, we provide evidence for this picture by examining the physics of a scalar field in de Sitter space, explaining how the dS isometries are realized on it and how mode solutions in the global and static patches are related. Sec. 5 concludes the paper by discussing open problems and future directions.

2 Bulk/boundary maps and de Sitter holography

In the spirit of the well known AdS/CFT correspondence [34, 35], one could try to set up a correspondence between global de Sitter space and a Euclidean field theory on the de Sitter boundaries. Specifically, the GKPW dictionary [35] states that (for Euclidean AdS)

$$\langle \exp(\int_{bnd} \phi_0 O) \rangle = Z_{bulk}(\phi_0) \quad (2)$$

where Z is the bulk string partition function as a functional of boundary data (or $Z_{bulk}(\phi_0) = \exp(-S_{bulk}(\phi))$ in the supergravity limit), ϕ_0 is the value of a bulk field ϕ at the spatial boundary of AdS, and O is an operator in the dual CFT. This precise formula also implies that the meaningful observables for string theory in AdS space are given by the correlation functions of the dual CFT. In Lorentzian signature further subtleties are introduced in which non-fluctuating non-normalizable modes implement the boundary conditions on AdS and correspond to field theory sources, while fluctuating normalizable modes in AdS space correspond to field theory states [36, 37]. The nature of the boundary CFT is determined by the fact that the near horizon geometry of certain solitonic states of string theory includes an AdS factor [34]. In the case of de Sitter space, the absence of a direct argument for a duality from the physics of string solitons makes the discussion more tentative; nevertheless, it is interesting to see what one gets.

Given the intuitions arising from the AdS/CFT correspondence and the fact that the early and late time infinities of de Sitter \mathcal{I}^\pm are natural holographic screens [38], can one write a de Sitter analog of (2)? If so, one would attempt to write the following formula

$$\langle \exp(\int_{I^-} \phi^{\text{in}} O^{\text{in}} + \int_{I^+} \phi^{\text{out}} O^{\text{out}}) \rangle = Z_{bulk}(\phi^{\text{in}}, \phi^{\text{out}}) \quad (3)$$

where $Z_{bulk}(\phi_{\text{in}}, \phi_{\text{out}})$ is the lorentzian bulk path integral for given the initial and final values of the bulk field ϕ .

If we want to interpret the left hand side in (3) as the generating function of correlators in some conventional euclidean CFT, equation (3) apparently involves two mismatched quantities. On one side we have a real quantity (given the real value of the generating functional) and on the other side we have a complex valued quantity - the lorentzian bulk path integral. The mismatch may point to a possible problem with unitarity in the dual CFT. Another interpretation of the mismatch is that the dual Hilbert space is equipped with a non-canonical Hilbert space structure. We will find evidence for this in later sections.

This mismatch, however, is just one of the problems with specifying (3). Recall that in the Lorentzian AdS/CFT correspondence, the boundary behaviour of fields is specified by the non-normalizable modes which grow near the AdS boundary. Normalizable modes, which decay near the boundary, correspond to fluctuating states, and giving them a classical background value turns on VEVs for composite operators

in the dual CFT [36, 37]. Fixing the non-normalizable mode corresponds to choosing a background, or, in the path integral language, picking boundary conditions for the path integral so that the fields approach a certain solution to classical equations of motion near the boundary.

By contrast, in de Sitter space there is no such distinction between normalizable and non-normalizable modes – all mode solutions are normalizable. There is a range of masses ($0 < 4m^2l^2 < d^2$, in $d + 1$ dimensional de Sitter, with cosmological scale l) in which a basis may be chosen in which the fields have a scaling growth or decay in time, but the solutions are still normalizable in the Klein-Gordon norm. For larger masses we can pick a basis of modes that show oscillatory behaviour with a fixed frequency near infinity [4, 7, 39, 40]. We will refer to these scaling/fixed frequency solutions at early and late time infinity as $\phi_{\text{in,out}}^\pm$. All of this is reminiscent of the situation in AdS for masses in the range $-d^2 < 4m^2l^2 < 0$ (in $d + 1$ dimensional AdS, with cosmological scale l [41, 36]), where all mode solutions for scalar fields are normalizable. In these cases it is necessary to pick one scaling behaviour as dual to sources and the other as dual to VEVs [36, 37]. While we might try the same strategy in de Sitter space, the fact that the classical equations of motion are second order implies that for classical solutions we can either specify both the “source” and the “VEV” scaling behaviour at one de Sitter boundary (e.g., ϕ_{in}^\pm), or we can specify the “source” scaling behaviour at both de Sitter boundaries (e.g., $\phi_{\text{in,out}}^+$ as in (3)). The latter description is better suited for a path integral setup as in (3). In effect, the sources at \mathcal{I}^+ determine the VEVs at \mathcal{I}^- and vice-versa.

The authors of [7, 40] address the same issue by considering four CFT operators $O_{\text{in,out}}^\pm$ that are related to each of the boundary behaviours described above. In particular [40] proposes a method for calculating the correlation functions of such CFT operators from a de Sitter analog of the boundary S-matrix calculations of AdS space [42]. If there is a bulk-boundary relationship in de Sitter space there can only be two independent operators of this kind, so, as discussed in [40] there is a relation between $O_{\text{in,out}}^\pm$ on shell. Consequently, it is not entirely clear how the generating functional of the correlators of these operators computed by the prescription in [40] can be related to a natural de Sitter bulk quantity like the path integral.

Another outstanding issue is whether the left hand side of (3) should be regarded as a theory that lives on a single sphere, or to a theory that lives on two separate spheres. In [4] it was argued that the theory dual to de Sitter should live on a single sphere in view of the antipodal singularities of propagators in de Sitter space. Certainly, for free fields, the values of the fields at \mathcal{I}^- are directly expressed in terms of the values of the fields at \mathcal{I}^+ , and it seems reasonable to view everything as living on a single sphere. However, once we turn on interaction, and turn to the full quantum gravitational path integral, the relation between the two boundaries becomes much less manifest. For instance, one can put two different metrics on the two boundaries. We will indeed find indications that the dual theory lives on two separate spheres.

In [3] Witten suggested that the bulk lorentzian path integral could be naturally associated to an inner product of two states in a complex Hilbert space of the dual theory (given a suitable action Θ of the CPT symmetry group). Given the bulk path integral

$$\langle f|i\rangle = Z_{bulk}(\phi_i, \phi_f) \quad (4)$$

the inner product $(,)$ is defined via $(f, i) = \langle \Theta f|i\rangle$. (The CPT transformation here maps the "out" states into "in" states allowing us to construct an inner product for the latter.) This suggests that perhaps a de Sitter holographic dictionary (if it exists) would be defining a Hilbert space structure and not an S-matrix, although [3] does propose a method to relate the de Sitter lorentzian path integral to local correlation functions of a dual field theory. Still, the precise definition of the right hand side of (4) runs into the same problems as we discussed above.

Given these different interpretations and possibilities, in the remainder of this section we explore aspects of the physics of de Sitter that are relevant for holography, either by making sense of (3) or by a suitable modification of this equation. Specifically, we review harmonic analysis on de Sitter space, and point out that there is an interesting non-local map from Lorentzian de Sitter space to Euclidean anti-de Sitter space. This map commutes with the action of the isometry group, and therefore maps solutions of free field equations into solutions. Such a non-local map suggests the interesting possibility of understanding de Sitter holography using the well known bulk/boundary dictionary from the AdS/CFT duality. Next, we discuss the on-shell action for free scalar fields, and point out the problems with the application of the standard GKPW-like machinery [35] to de Sitter space. After that, we turn to gravity in $2+1$ dimensions with a positive cosmological constant and display a family of solutions in which equal time sections are arbitrary genus Riemann surfaces. We consider the on-shell action of these spaces and again find subtleties with the standard GKPW approach. For example, the conformal anomaly contributions from the surfaces at \mathcal{I}^\pm cancel against each other. Finally, we discuss some aspects of the $2+1$ dimensional case from the Chern-Simons theory perspective. A basic conclusion from these studies is that if there is a dual to de Sitter, it probably lives on two spheres rather than one, and that a GKPW-like formula such as (3) cannot be naively applied.

2.1 A non-local map from dS to AdS

To gain insight into holography in de Sitter space it is interesting to ask whether there is a map from the physics of Lorentzian de Sitter to Euclidean AdS in view of (1). Such a map cannot amount to mere analytic continuation since the scalar Green function on AdS does not continue to a 2-point function in a de Sitter vacuum [7]. Also, since de Sitter space has two boundaries while Euclidean de Sitter has only one both dS boundaries must map into the single AdS one. As a preliminary step we show below that there is an interesting nonlocal map that maps solutions to free

wave equations in dS and EAdS onto each other while preserving the isometries of the hyperboloid (1). Note that the non-locality implies that this map cannot be used to transform the local physics of AdS or its dual into the local physics of dS or its dual.

Some of the analysis below is carried out in three dimensions since it is convenient since to use the representation theory of $SL(2, \mathbb{C})$; higher dimensional extensions of the results should be straightforward.

2.1.1 Harmonic Analysis

The harmonic analysis of dS_3 is discussed in chapter 6, section 4 of [43]. There, the decomposition of $L^2(dS_3)$ in terms of irreducible representations of $SL(2, \mathbb{C})$ is given. To describe this result, we start with the description of lorentzian de Sitter space as a hyperboloid immersed in the flat Minkowski space. de Sitter space is described by

$$P(X, X) = 1 \quad (5)$$

where

$$P(X, Y) = -X_0 Y_0 + X_1 Y_1 + \dots + X_d Y_d \quad (6)$$

represents a Minkowski signature quadratic form (we have set the cosmological constant to 1). The decomposition of a function $f(X)$ in $L^2(dS_3)$ reads

$$\begin{aligned} f(X) = & \frac{1}{2(4\pi)^3} \int_0^\infty \rho^2 d\rho \int_{S^2} F(U(\omega); \rho) |P(X, U(\omega))|^{-\frac{1}{2}i\rho-1} d\omega \\ & + \frac{4}{\pi^2} \sum_{n=1}^\infty n \int_{S^2} F(U(\omega), Y; 2n) e^{2in\theta(Y, X)} \delta(P(U(\omega), X)) d\omega \end{aligned} \quad (7)$$

where $\cos \theta(Y, X) = -P(X, Y)$, Y is any point on dS_3 , and $U(\omega)$ is a family of points on the null cone $P(U, U) = 0$ which are parametrized by $\omega \in S^2$. If we write ω as (U_1, U_2, U_3) with $U_1^2 + U_2^2 + U_3^2 = 1$, then $U(\omega) = (1, U_1, U_2, U_3)$. Thus $U(\omega)$ sweeps out a two-sphere on the forward null cone. Finally, $d\omega$ is the standard measure on the two-sphere. The two-sphere can be replaced by any other two-surface homeomorphic to it with a suitable change in the measure $d\omega$. The Fourier modes labeled by F can be obtained from $f(X)$ via suitable integrals of $f(X)$ and are independent of Y , see [43] for details.

The first term in (7) contains principal series representations of $SL(2, \mathbb{C})$, similar to what one finds in the case of euclidean AdS. In the latter case, one typically analytically continues the functions $|P(X, U(\omega))|^{-\frac{1}{2}i\rho-1}$ to imaginary values of ρ , in which case these functions become the wave functions discussed in e.g. [44, 45]. At the same time, they are also equal to the bulk-boundary propagator introduced by Witten [35].

To make the last statement explicit, we express $|P(X, U)|$ in terms of coordinates on the inflationary patch. In Euclidean AdS, we parametrize X as

$$X = (\frac{1}{2}(e^t + e^{-t} + e^t z_i z^i), \frac{1}{2}(e^t - e^{-t} - e^t z_i z^i), e^t z_1, \dots, e^t z_{d-1}) \quad (8)$$

from which we obtain the metric $ds_{\text{EAdS}}^2 = dt^2 + e^{2t}(dz_0^2 + \dots + dz_{d-1}^2)$ with t as the radial direction, z_0 as Euclidean time and the AdS boundary at $t \rightarrow \infty$, in which limit the EAdS hyperboloid approaches the null surface $P(U, U) = 0$. The conformal boundary of EAdS is therefore a sphere embedded in this null asymptote. Conformal rescaling of the boundary sphere moves it along the null cone $P(U, U) = 0$. Therefore writing points on the (conformally rescaled) boundary sphere as

$$U = (\frac{1}{2}(1 + y_i y^i), \frac{1}{2}(1 - y_i y^i), y_1, \dots, y_{d-1}), \quad (9)$$

we obtain

$$|P(X, U)|_{\text{EAdS}} = \frac{1}{2}(e^{-t} + e^t(y - z)^2). \quad (10)$$

Raising this to the suitable power indeed gives the bulk-boundary propagator of euclidean AdS. To get Lorentzian de Sitter space, we replace the term e^{-t} by $-e^{-t}$ in X^0 and X^1 in the parametrization (8). This gives the metric $ds^2 = -dt^2 + e^{2t}d\vec{z}^2$ which covers half of de Sitter (the inflating patch). As $t \rightarrow \infty$ the \mathcal{I}^+ the de Sitter hyperboloid approaches to the null surface $P(U, U) = 0$. The conformal boundary of de Sitter is therefore a sphere embedded in the cone $P(U, U) = 0$, just like the EAdS boundary parameterized in (9). Conformal rescaling of the boundary amounts to rescaling U . With these parametrizations we find that

$$|P(X, U)|_{dS} = \frac{1}{2}|-e^{-t} + e^t(y - z)^2|. \quad (11)$$

Again, raising this to the power in (7) gives the bulk-boundary propagator in de Sitter space (see [40] for a discussion using conformal time for the inflationary patch).

Actually, the bulk-boundary propagator $|P(X, U)|^{-1-i\rho/2}$ is not well-defined as it stands, because $P(X, U)$ changes sign as a function of t . We can define it once we choose a branch cut above or below the real t -axis in the complex t -plane. If we choose the branch cut above the t -axis, the functions $|P(X, U)|^{-1-i\rho/2}$ are like the Euclidean modes in de Sitter space, because they are analytic on the lower hemisphere of the Euclidean sphere that is obtained by the analytic continuation of dS space.

It is straightforward to express everything in global coordinates for de Sitter with metric $ds^2 = -dt^2 + \cosh^2 t d\Omega^2$. We can take e.g. $X = (\sinh t, \mathbf{n} \cosh t)$ and $U = (1, \mathbf{m})$ with \mathbf{m}, \mathbf{n} unit vectors in \mathbf{R}^d . Then

$$|P(X, U)|_{dS} = |- \sinh t + (\mathbf{n} \cdot \mathbf{m}) \cosh t|. \quad (12)$$

The second term in (7) is interesting because it does not have a counterpart in AdS. The Fourier components $F(U(\omega), Y; 2n)$ are obtained from the integral of $f(X)$

along null geodesics. A null geodesic in de Sitter space is given by a straight line $Y + tU$, where Y is a point on de Sitter space, so that $P(Y, Y) = 1$, and U is a point on the cone $P(U, U) = 0$ that satisfies $P(Y, U) = 0$.

In summary, the decomposition of $L^2(dS_3)$ is quite similar to the decomposition of $L^2(EAdS_3)$, except that $L^2(dS_3)$ contains an extra set of representations associated to the integrals of functions along null geodesics.

2.1.2 A non-local map from dS to AdS

It has been suggested that quantum gravity on de Sitter space is dual to a single Euclidean CFT [4]. On the other hand, in the standard AdS/CFT correspondence, Euclidean CFT's are dual to quantum gravity on Euclidean Anti-de Sitter space. Thus it is worth exploring whether there exists a relation between de Sitter space and Anti-de Sitter space that is more subtle than analytic continuation. To find such a relation, we will now look for a (possibly non-local) map from de Sitter space to anti-de Sitter space that commutes with the isometry group $SO(d, 1)$ that de Sitter and Euclidean Anti-de Sitter have in common.

We consider a non-local map to of the form

$$\psi(Y) = \int dX K(Y, X) \phi(X) \quad (13)$$

where dX denotes the invariant measure on dS_d , and demand that the kernel $K(X, Y)$ has the fundamental property that it commutes with the $SO(d, 1)$ actions. In other words,

$$K(gX, gY) = K(X, Y) \quad (14)$$

for $g \in SO(d, 1)$.

Using suitable g , we can always achieve that $X_i = 0$ for $i < d$, and $X_d = 1$. Call this point E . Then it is sufficient to know $K(E, Y)$, because by acting with the group we can recover the rest of K from this.

The point E is preserved by a $SO(d-1, 1)$ subgroup. This can be used to put Y in the form

$$Y(\xi) \equiv (Y_0, \dots, Y_d) = (\sqrt{1 + \xi^2}, 0, \dots, 0, \xi). \quad (15)$$

The group action cannot be used to change the value of ξ . Thus, the kernel is completely determined by a function of a single variable,

$$K(E, Y(\xi)) \equiv K(\xi). \quad (16)$$

It is easy to write down the kernel explicitly, once we are given $K(\xi)$. Clearly $P(gX, gY) = P(X, Y)$, and we also observe that $P(E, Y(\xi)) = \xi$. Therefore,

$$K(X, Y) = \int d\xi \delta(P(X, Y) - \xi) K(\xi) \quad (17)$$

satisfies all the required properties, it is group invariant and reduces to $K(\xi)$ for $X = E, Y = Y(\xi)$. The simplest form of the kernel is $K(\xi) = \delta(\xi)$. In this case our transform is closely related to the Radon transform [46] known in the study of tomography.

We now investigate the transform for a free massive scalar. In a slightly different notation from the one before, we label points on the $d - 1$ sphere by unit normals \mathbf{n}, \mathbf{m} . In general the transform is

$$\psi(\rho, \mathbf{m}) = \int d\xi \int dt d^{d-1} \mathbf{n} \cosh^{d-1} t K(\xi) \delta(-\sinh t \cosh \rho + \cosh t \sinh \rho (\mathbf{n} \cdot \mathbf{m}) - \xi) \phi(t, \mathbf{n}). \quad (18)$$

Here, we parametrized a point on dS as $X = (\sinh t, \mathbf{n} \cosh t)$ and a point on $EAdS$ as $Y = (\cosh \rho, \mathbf{m} \sinh \rho)$ ($\rho \geq 0$) giving the metrics $ds_{\text{dS}}^2 = -dt^2 + \cosh^2 t d\Omega^2$ and $ds_{\text{EAdS}}^2 = d\rho^2 + \sinh^2 \rho d\Omega^2$. The dS boundaries are at $t \rightarrow \pm\infty$ and the EAdS boundary is at $\rho \rightarrow \infty$.

It is not so easy to do this integral for a global mode solution, because we have to integrate spherical harmonics and hypergeometric functions, and the result is a complicated linear superposition of modes in EAdS. To get some idea of what this transform does, we apply it to a Green's function $G(t, \mathbf{n}, t_0, \mathbf{n}_0)$ that obeys

$$\Delta G = \delta(t - t_0) \delta(\mathbf{n} - \mathbf{n}_0). \quad (19)$$

The transform of G obeys

$$\Delta \psi = K(-\sinh t_0 \cosh \rho + \cosh t_0 \sinh \rho (\mathbf{n}_0 \cdot \mathbf{m})). \quad (20)$$

Thus, even though we start with a localized source for the Green's function, after the transform we find a source that is smeared out over all of $EAdS$.

In case $K(\xi) = \delta(\xi)$, we get

$$\Delta \psi = \delta(-\sinh t_0 \cosh \rho + \cosh t_0 \sinh \rho (\mathbf{n}_0 \cdot \mathbf{m})). \quad (21)$$

The right hand side contributes on the surface

$$\tanh \rho = \frac{\tanh t_0}{(\mathbf{n}_0 \cdot \mathbf{m})} \quad ; \quad \text{sign}(t_0) = \text{sign}(\mathbf{n}_0 \cdot \mathbf{m}). \quad (22)$$

The condition on the sign arises because $\rho \geq 0$ and we see that antipodal points in de Sitter ((t_0, \mathbf{n}_0) and $-(t_0, \mathbf{n}_0)$) get mapped to the same points in Euclidean AdS (ρ, \mathbf{m}) that solves (22). If $t_0 > 0$, ρ takes it smallest value at $\rho = t_0$, while for $t_0 < 0$, ρ has as smallest value $|t_0|$, when $\mathbf{m} = -\mathbf{n}_0$. The de Sitter boundaries at $t \rightarrow \pm\infty$ are mapped into the EAdS boundary at $\rho \rightarrow \infty$.

Although the map is non-local, equation (22) shows that the support of $\Delta \psi$ is on a surface with $\rho \leq |t_0|$. Thus, once we send $|t_0|$ off to infinity, ρ should go to infinity as well. Therefore we expect that the bulk-boundary propagator should have

a simple transform. As an example we work out the $d = 3$ case where this propagator is given by $|P(X, U)|^{-1-i\rho/2}$. Here X is a point in de Sitter space ($P(X, X) = 1$) and U is a point in the conformal boundary of de Sitter space specified by $P(U, U) = 0$ as discussed earlier. Hence, in transforming from de Sitter to EAdS, we map X into points Y in EAdS, but U is simply a parameter. The transform (with $K(\xi) = \delta(\xi)$) is given by

$$\psi(Y, U) = \int \delta(P(X, Y)) |P(X, U)|^{-1-i\rho/2} dX \quad (23)$$

with $Y \in EAdS$ and U a point on the cone $P(U, U) = 0$. Since the conformal boundary of EAdS also lies in the cone $P(U, U)$, we can treat U equally as parametrizing boundary points in EAdS or de Sitter (the latter subject to an antipodal identification as described above). Since $\psi(Y, U)$ satisfies $\psi(Y, U) = \psi(gY, gU)$ for $g \in SO(3, 1)$ we can without loss of generality take $Y = (1, 0, 0, 0)$ and $U = (\xi, \xi, 0, 0)$. If we parametrize $X = (\sinh t, \cosh t \mathbf{n})$ with $\mathbf{n} = (n_1, n_2, n_3)$ a point on the unit two-sphere, the right hand side of (23) becomes

$$\int dt d^2 \mathbf{n} \cosh^2 t \delta(-\sinh t) |-\xi \sinh t + \xi \cosh t n_1|^{-1-i\rho/2}. \quad (24)$$

This is proportional to $|\xi|^{-1-i\rho/2}$. As ξ can be invariantly written as $P(Y, U)$, we conclude that $\phi(Y, U)$ is proportional to $P(Y, U)^{-1-i\rho/2}$. Therefore, the non-local map takes the bulk-boundary propagator of dS into the bulk-boundary propagator of EAdS. Since solutions to the free wave equations in EAdS and in de Sitter can be constructed by convolving the bulk-boundary propagator with boundary data, we see that such solutions map onto each other.

Note that in the AdS/CFT correspondence the parameter $i\rho$ here is related to the conformal dimension of the dual operator and real (imaginary) ρ corresponds to an imaginary (real) conformal dimension associated with fields below (above) the Breitenlohner-Freedman mass bound for AdS [41]. In de Sitter space it has been noted that there is a sort of inversion of the Breitenlohner-Freedman bound – fields with masses above a certain bound would map onto operators with imaginary conformal dimension in a dual CFT while lower masses would map to operators with real dimension [4]. Here the reality of the conformal dimension is being preserved in the dS-EAdS map. If we want to have real dimensions on both sides of the map we must stick to fields in de Sitter with masses lower than the bound in [4]. However, as we will discuss later in this paper there may be ways of interpreting apparent imaginary conformal dimensions in CFTs with different hermiticity conditions from the usual ones.

Altogether we have constructed a non-local map from dS to EAdS that commutes with the isometries, maps the two boundaries of dS onto the single boundary of EAdS, using an anti-podal identification for one of the two, and maps bulk-boundary propagators to bulk-boundary propagators. It therefore is tempting to define a dS/CFT correspondence by first mapping everything to EAdS, and subsequently applying the

standard rules of AdS/CFT. This certainly yields a single Euclidean CFT. However, since the map is non-local, it does not map local interactions of fields to local interactions of fields, and does not naively extend to a well-defined map on the level of (super)gravity. More importantly, the map has a kernel [43], which is similar to the second line in (7). If we know the field on EAdS, we cannot reconstruct the integrals of the field on dS along null geodesics. That information is lost under this map. We view this as suggestive evidence that it is not sufficient to describe dS space in terms of a single CFT. We will later advocate a picture where the dual of dS involves two (entangled) conformal field theories.

It would be interesting further to study this non-local map, and to find out whether it allows us to set up a GKPW-like formulation for de Sitter space, and to extend this map to other time-dependent backgrounds.

2.2 On-shell action-scalar fields

To make sense of expressions of the form (3) we need to define the left hand side via the right hand side. Naively, as in AdS/CFT, this is the on-shell value of the (super)gravity action evaluated with given boundary conditions. At tree-level, this becomes the (super)gravity action evaluated on a solution of the equations of motion. For a free scalar field with action

$$S = \int_M d^d x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2) \quad (25)$$

the action reduces for a solution of the equations of motion to

$$S = - \int_{\partial M} d^{d-1} x \sqrt{\hat{g}} \phi \partial_t \phi = - \int_{\mathcal{I}^+} \sqrt{\hat{g}} \phi \partial_t \phi + \int_{\mathcal{I}^-} \sqrt{\hat{g}} \phi \partial_t \phi \quad (26)$$

when we choose a de Sitter metric of the form

$$ds^2 = -dt^2 + \hat{g}_{ij} dx^i dx^j. \quad (27)$$

As discussed earlier, all solutions to the scalar field equations are normalizable, hence in order to impose boundary conditions in the path integral in might be necessary to add boundary terms to the actions in (25) and (26). It is easy to show that as $t \rightarrow -\infty$, solutions to the field equations behave as [39, 7, 40]

$$\phi \sim \phi_{\text{in}}^+ e^{2h^+ t} + \phi_{\text{in}}^- e^{2h^- t} \quad (28)$$

and as $t \rightarrow +\infty$ as

$$\phi \sim \phi_{\text{out}}^+ e^{-2h^+ t} + \phi_{\text{out}}^- e^{-2h^- t}. \quad (29)$$

The scaling dimensions are

$$2h^\pm = \frac{d-1}{2} \pm \sqrt{\frac{(d-1)^2}{4} - m^2} \quad (30)$$

and we will assume that we are in the regime $4m^2 > (d-1)^2$ so that h^\pm are complex². In (26) there are contributions $\phi^+\dot{\phi}^+$, $\phi^+\dot{\phi}^-$, $\phi^-\dot{\phi}^+$ and $\phi^-\dot{\phi}^-$ at each boundary. The first and the last of these scale as $\exp(\pm 4h^\pm t)$.

In standard AdS/CFT correspondence we similarly get a $\phi \partial_\rho \phi$ boundary action (where ρ is the radial direction away from the boundary). We rewrite this as a nonlocal expression in terms of boundary data by using a bulk-boundary propagator, and regulate by adding local counterterms to remove divergences. For a scalar field such counterterms can be built out of the boundary value of the field and its covariant derivatives with respect to the boundary coordinates. The surviving finite terms yield CFT correlation functions. The multi-local terms give n-point functions at separated points and the local parts are associated with contact terms and one-point functions. In particular, in Lorentzian signature, local terms arising from $\phi^+\phi^-$ type terms in the language used above lead to one-point functions in a CFT state determined by the ϕ^+ scaling piece of the bulk solution. In de Sitter space, with $4m^2 > (d-1)^2$, this strategy cannot be applied directly. In (28,29) we have only specified the leading scaling behaviour of mode solutions. After accounting for the boundary metric, the leading and subleading terms in the $\phi^+\dot{\phi}^+$ and $\phi^-\dot{\phi}^-$ contributions to the boundary action, as well as in possible local counterterms, scale as $\exp(\pm(4h^\pm - 2n)t)$ for integer n . Since the square root in (30) is imaginary this means that there are no terms arising from $\phi^\pm\dot{\phi}^\pm$ contributions that approach a finite and well-defined limit as they approach the de Sitter boundary – they either oscillate or vanish after regulation. In conformal time as opposed to the global time used above the oscillations will be wild, growing faster as the boundary is approached. It seems natural to remove any such wildly oscillating behavior, in which case nothing is left. (When $4m^2 < (d-1)^2$ these oscillations do not arise, but other issues are relevant in that case.)

The terms with a finite limit as we approach the boundary are $\phi^+\dot{\phi}^-$ and $\phi^-\dot{\phi}^+$. In the Euclidean AdS/CFT case, ϕ^+ and ϕ^- are not independent, but one is given in terms of the other by requiring that the solution is regular in the interior of the Euclidean AdS space, while in the Lorentzian case ϕ^+ is independent and related to the specification of a CFT state. In either case, we obtain non-trivial finite contributions from this term, related to contact terms (which we may wish to remove) or to one-point functions in a CFT state. In the case of Lorentzian de Sitter space, ϕ^+ and ϕ^- are independent. Actually, they are each others complex conjugate, and it is really the real and imaginary part of ϕ^+ that are independent. The on-shell action

²For $4m^2 < (d-1)^2$ the weights h^\pm are real and the situation in de Sitter space is more closely related to the situation in Euclidean AdS with growing and decaying modes at the boundaries. Nevertheless, many of the problems we discuss here are not resolved by restricting attention to real h^\pm only. Likewise, as discussed in the previous section, the solutions to the wave equation are normalizable for any value of m^2 .

becomes

$$S = (d-1) \int_{I^-} d^{d-1}x \sqrt{\hat{g}} \phi_{\text{in}}^+ \phi_{\text{in}}^- - (d-1) \int_{I^+} d^{d-1}x \sqrt{\hat{g}} \phi_{\text{out}}^+ \phi_{\text{out}}^- \quad (31)$$

where the factor of $d-1$ arises from the time derivatives, $d-1 = 2(h^+ + h^-)$. Earlier we discussed that we might follow the AdS/CFT philosophy and associate operators and VEVs with ϕ^\mp scalings or, alternatively, following [7, 40] we could try to associate different CFT operators O^\mp with these scalings. In these two cases (31) would be associated with VEVs of CFT operators or with contact terms between operators respectively. In addition, the two terms in (31) are not independent, but are proportional to each other, because ϕ_{out}^\pm are linear combinations of ϕ_{in}^\pm for a solution of the classical equations of motion.

Thus, the naive on-shell action for a free scalar field merely gives a contact term. It is an interesting question whether there are other ways to define the supergravity action, which yields results that resemble the AdS/CFT answer and produces CFT two-point functions. One direction is to use the bulk-boundary propagator and to add suitable non-local boundary terms to the action, for instance as in [47]. Normally, we do not add such a non-local quantities – while local counterterms may be used to remove divergences in an effective action, non-local terms modify the correlation functions. However, nonlocal boundary terms may be acceptable if they arise as a method of imposing boundary conditions in de Sitter space that are necessary for the validity of a correspondence with a dual CFT. We seek to impose boundary conditions on a free scalar field that lead to a non-trivial action for solutions to the equations of motion. Given the asymptotic behavior of the fields,

$$\phi \sim \phi_{\text{in}}^+ e^{2h^+t} + \phi_{\text{in}}^- e^{2h^-t} \quad (32)$$

one notices that

$$\lim_{t \rightarrow -\infty} e^{-2h^+t} (\partial_t - 2h^-) \phi = 2(h^+ - h^-) \phi_{\text{in}}^+ + \text{subleading}. \quad (33)$$

Therefore, fixing the left hand side of this equation imposes a suitable boundary condition on ϕ . At the same time, a general solution of the equations of motion is of the form

$$\phi(t, \mathbf{m}) = \int d\mathbf{n} \rho(\mathbf{n}) |-\sinh t + \cosh t(\mathbf{n} \cdot \mathbf{m})|^{-2h^+}. \quad (34)$$

If we define $\mathbf{A}(\rho)$ via

$$\mathbf{A}(\rho) = e^{-2h^+t} (\partial_t - 2h^-) \int d\mathbf{n} \rho(\mathbf{n}) |-\sinh t + \cosh t(\mathbf{n} \cdot \mathbf{m})|^{-2h^+} \quad (35)$$

then the idea, following [47] is to add a boundary term that imposes the boundary condition

$$e^{-2h^+t} (\partial_t - 2h^-) \phi = \mathbf{A}(\rho). \quad (36)$$

The relevant non-local boundary term is proportional to $\int \phi A(\rho)$, and when evaluated on classical solutions of the equation of motion yields the CFT two-point function for ϕ^+ , at least in the case of the AdS/CFT correspondence. In the dS/CFT case, there are some extra complications. First, one has to be quite careful in defining integrals like (34), which have a singularity, and second, one has to impose the boundary conditions on both sides simultaneously. We have not studied this in detail, but it would be nice to understand this better.

Yet another approach to define correlation functions from supergravity is the S-matrix inspired approach of [4, 7, 40]. In this approach, each of the scaling behaviours ϕ_{in}^{\pm} of bulk fields is associated with the insertion of a CFT operator on \mathcal{I}^+ or \mathcal{I}^- , and a method inspired by the boundary S-matrix for AdS space was proposed for calculating the correlation functions of operators. CFT-like two point functions and contact terms are obtained. However, unlike the AdS case, it is not clear how such de Sitter calculations are related to a bulk quantity with a classical limit like the path integral. In particular since ϕ_{in}^{\pm} are not all independent of each other classically, an arbitrary correlation function of corresponding operators $O_{\text{in,out}}^{\pm}$ cannot be computed from any quantity that has a classical limit in the spacetime theory. Indeed in the free field theory, if we tried to associate the correlators of $O_{\text{in,out}}^{\pm}$ with a bulk path integral as

$$\langle \exp(\int_{\mathcal{I}^-} \phi_{\pm}^{\text{in}} O_{\mp}^{\text{in}} + \int_{\mathcal{I}^+} \phi_{\pm}^{\text{out}} O_{\mp}^{\text{out}}) \rangle = Z_{\text{bulk}}(\phi_{\pm}^{\text{in}}, \phi_{\pm}^{\text{out}}). \quad (37)$$

then the computation on the right hand side is dominated by a saddlepoint only when we restrict ϕ to solutions to the equations of motion. In this case, the proposed CFT operators $O_{\text{in,out}}^{\pm}$ that couple to ϕ_{in}^{\pm} satisfy, the property that

$$\phi_{\text{in}}^+ O_{\text{in}}^- + \phi_{\text{in}}^- O_{\text{in}}^+ = \phi_{\text{out}}^+ O_{\text{out}}^- + \phi_{\text{out}}^- O_{\text{out}}^+ \quad (38)$$

if ϕ is a solution of the equation of motion. Then if as in [4, 7, 40] we associate all of these operators with a single sphere, the answer becomes $Z = 1$. If the operators are associated with different spheres, the answer becomes more interesting.

In the next section, we will see that the above observations also apply to some extent once we look at the Einstein-Hilbert part of the action. At this point, there could be two points of view. One could take the point of view that the naive supergravity action which yields just a contact term is the right answer. This would be very close to the proposal of Witten [3] because the contact term is closely related to the inner product in the free field theory on de Sitter space, and free field action seems to compute nothing but this inner product. In this case, all the interesting physics would be hidden in interactions and in higher order effects. On the other hand, a much richer structure is obtained using the S-matrix approach of [4, 7, 40]. As we will explain later, these two points of view do not necessarily have to disagree with each other. Our final proposal for a dual description of de Sitter space will involve two CFT's living on two different boundaries. These two theories are entangled. From

the S-matrix point of view, we are computing correlation functions in these entangled CFT's. However, one might equally well argue that the set of correlation functions in one of the two CFT's represent the wave functions of the Hilbert space of the theory (very much as in the relation between Cherns-Simons theory and CFT), and the correlation functions in the entangled state represent the inner product on this Hilbert space. Thus, our final proposal incorporates both points of view in a satisfactory manner, though in order to probe the full structure the S-matrix point of view is needed.

2.3 On-shell action-gravity

In this subsection we investigate the gravitational on-shell bulk action of de Sitter space. We start by describing “new” classical solutions of 2+1 dimensional gravity with a positive cosmological constant. The solutions we find are very similar to the solutions of 2+1 gravity with a negative cosmological constant, that were found in the Fefferman-Graham approach in the context of the AdS/CFT duality, see e.g. [48], [49] and [50].

The most general solution reads

$$ds_{I^+}^2 = -dt^2 + e^{2t}e^\phi dw d\bar{w} - \frac{1}{2}(T^\phi - T(w))dw^2 - \frac{1}{2}(\bar{T}^\phi - \bar{T}(\bar{w}))d\bar{w}^2 - Rdwd\bar{w} \\ + \frac{1}{4}e^{-2t}e^{-\phi}((T^\phi - T(w))dw + Rd\bar{w})(\bar{T}^\phi - \bar{T}(\bar{w}))d\bar{w} + Rdw), \quad (39)$$

where

$$T^\phi = \partial_w^2 \phi - \frac{1}{2}(\partial_w \phi)^2, \quad \bar{T}^\phi = \bar{\partial}_{\bar{w}}^2 \phi - \frac{1}{2}(\bar{\partial}_{\bar{w}} \phi)^2, \quad R = \partial_w \bar{\partial}_{\bar{w}} \phi. \quad (40)$$

Here $T(w)$ and $\bar{T}(\bar{w})$ are (anti)holomorphic quadratic differentials. The metric at I^+ is conformal, $ds^2 = e^\phi dw d\bar{w}$; Up to the finite number of free parameters contained in $T(w), \bar{T}(\bar{w})$ the metric at I^- is completely determined in terms of that on I^+ as

$$ds_{I^-}^2 = \frac{1}{4}e^{-\phi}((T^\phi - T(w))dw + Rd\bar{w})(\bar{T}^\phi - \bar{T}(\bar{w}))d\bar{w} + Rdw. \quad (41)$$

This is presumably a feature of $d = 2 + 1$ dimensions where the metric has no dynamical degrees of freedom. In higher dimensions one would expect to be able to independently fix the metric at I^+ and I^- .

We can get any Riemann surface of genus $g > 1$ by taking w, \bar{w} to live in the complex upper half plane and by modding out by a discrete subgroup $\Gamma \subset PSL(2, \mathbf{Z})$. This will work if the holomorphic quadratic differential is invariant under $\gamma \in \Gamma$,

$$T(\gamma(w)) \left(\frac{\partial \gamma}{\partial w} \right)^2 = T(w) \quad (42)$$

and

$$\phi(\gamma(w)) = \phi(w) - \log \left| \frac{\partial \gamma}{\partial w} \right|^2. \quad (43)$$

It is interesting to compare this to the most general solution with negative cosmological constant, which reads (in Euclidean space)

$$ds^2 = dt^2 + e^{2t} e^\phi dw d\bar{w} + \frac{1}{2}(T^\phi - T(w))dw^2 + \frac{1}{2}(\bar{T}^\phi - \bar{T}(\bar{w}))d\bar{w}^2 + Rdwd\bar{w} + \frac{1}{4}e^{-2t}e^{-\phi}((T^\phi - T(w))dw + Rd\bar{w})((\bar{T}^\phi - \bar{T}(\bar{w}))d\bar{w} + Rdw). \quad (44)$$

Note that this clarifies the relation between the Fefferman-Graham (FG) expansion in $\Lambda < 0$ and $\Lambda > 0$ [15, 11]. Given a solution to the Euclidean equations of motion, given in an FG expansion, we first send $e^{2t} \rightarrow e^{-2t}$, in other words we shift $t \rightarrow t + \pi i/2$. The metric will still solve the $\Lambda < 0$ equations of motion but no longer be Euclidean. Next we send $g_{\mu\nu} \rightarrow -g_{\mu\nu}$. This gives a Minkowski solution of the $\Lambda > 0$ solutions.

The general procedure is to start with a solution of the Minkowski equations of motion, and then to continue the coordinates in such a way so that the signature of space becomes $(+ - - - - - -)$. Next, we can send $g_{\mu\nu} \rightarrow -g_{\mu\nu}$ and we are done.

In particular, if we take $\phi = 0$, and $T(w) = \bar{T}(\bar{w}) = 0$ we get the inflationary patch. For $\phi = -2\log[1 + w\bar{w}]$ we get global de Sitter space,

$$ds^2 = -dt^2 + 4 \cosh^2 t \frac{dw d\bar{w}}{(1 + w\bar{w})^2}. \quad (45)$$

Similarly, when we insert this into the $\Lambda < 0$ equation we get global Euclidean de Sitter in the form

$$ds^2 = dt^2 + 4 \sinh^2 t \frac{dw d\bar{w}}{(1 + w\bar{w})^2}. \quad (46)$$

(Recall that $ds^2 = \frac{dw d\bar{w}}{(1 + w\bar{w})^2}$ is the metric on a two sphere, up to a factor of four.) In the Euclidean anti-de Sitter case, the metric is defined for $t \geq 0$ and $t = 0$ is a smooth point.

If we take $\phi = -2\log[(w - \bar{w})/i]$, we find in the de Sitter case

$$ds^2 = -dt^2 + 4 \sinh^2 t \frac{dw d\bar{w}}{(2\text{Im}(w))^2}, \quad (47)$$

and in the AdS case

$$ds^2 = dt^2 + 4 \cosh^2 t \frac{dw d\bar{w}}{(2\text{Im}(w))^2}. \quad (48)$$

This is the usual metric on the upper half plane that is $SL(2, Z)$ invariant, so once we divide by arbitrary finite subgroups of $PSL(2, Z)$ the same metric describes locally the dS (AdS) solution where the Riemann surface has genus $g > 1$. The AdS solution is well-known, and applies to any dimension, where the $g > 1$ Riemann surface is replaced by an Einstein metric with negative curvature (see e.g. [51]).

The de Sitter solution has a cosmological contraction/expansion singularity. It is not clear whether the AdS spaces with disconnected boundary can be made sense

of quantum mechanically. Witten and Yau claim [51] that the AdS theory does not make sense usually if the space on which the dual CFT lives has negative curvature (as it would in these cases). In the AdS₅/CFT₄ duality this is because the conformal coupling of the Ricci scalar to the scalars of the dual theory would cause the action to be unbounded from below. On the other hand, it makes perfect sense to consider a 1 + 1 dimensional CFT on a Riemann surface of arbitrary genus, and understanding its holographic dual might give important clues about a dS/CFT duality.

The w, \bar{w} part of the metrics can be written in matrix notation as

$$e^{2t+\phi} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm \frac{1}{2} e^{-2t-\phi} \begin{pmatrix} -R & -(\bar{T}^\phi - \bar{T}) \\ -(T^\phi - T) & -R \end{pmatrix} \right]^2 \quad (49)$$

with the plus sign being appropriate for de Sitter. For de Sitter, the metric degenerates at

$$2e^{2t} = e^{-\phi} (R \pm |T^\phi - T|), \quad (50)$$

and for AdS at

$$2e^{2t} = e^{-\phi} (-R \pm |T^\phi - T|). \quad (51)$$

The previous examples had $T^\phi = T = 0$. In general, we see that when the AdS solution is smooth the dS solution degenerates and the other way around.

Now we compute the on-shell action for these classical solutions. The Einstein action plus the extrinsic curvature term for the dS solutions $-dt^2 + h_{ij} dx^i dx^j$ is evaluated between $t = t_0$ and $t = t_1$ as

$$S = \frac{i}{16\pi G} \left\{ \int_{t_0}^{t_1} (-4\sqrt{h}) + [2\partial_t \sqrt{h}]_{t_0}^{t_1} \right\} \quad (52)$$

and for *AdS* case we obtain the same answer except that i is replaced by -1 . For dS, we also get that

$$\sqrt{h} = \frac{1}{8} e^{2t+\phi} ((2 - e^{-2t-\phi} R)^2 - e^{-2t-\phi} |T^\phi - T|^2) \quad (53)$$

and for AdS we need to send $R \rightarrow -R$. (Here the measure $dwd\bar{w}$ is taken to be $d\text{Re}(w)d\text{Im}(w)$.) We obtain

$$S = \frac{i}{32\pi G} \int dwd\bar{w} \left[2e^{2t_1+\phi} + 4Rt_1 + \frac{1}{2} e^{-2t_1-\phi} (|T^\phi - T|^2 - R^2) - 2e^{2t_0+\phi} - 4Rt_0 - \frac{1}{2} e^{-2t_0-\phi} (|T^\phi - T|^2 - R^2) \right]. \quad (54)$$

There are two divergences as $t_1 \rightarrow \infty$. These can be canceled by local counterterms, because one can easily show that $\sqrt{h} h^{ab} R(h)_{ab} = 4R = 4\partial\bar{\partial}\phi$.

We see that by adding boundary terms at both boundaries at $t = t_1$ and $t = t_0$ (which we imagine going to $\pm\infty$), the action becomes completely zero. (The same result was obtained by [8] following the procedure of [13, 14, 6, 5, 52].)

We would have naively expected the Liouville action to appear. That this does not happen may seem puzzling, because the Liouville action is needed to reproduce the conformal anomaly, and the arguments for the conformal anomaly given e.g. in [12] appear to be valid for de Sitter as well. However, a more precise inspection shows that this is not exactly true. The conformal anomaly arises from an on-shell action like (54) from changing $\phi \rightarrow \phi + \epsilon$ and $t \rightarrow t - \epsilon/2$. Under this change of coordinate, we pick up a contribution $\int 4\epsilon R$ from I^+ , but at the same time a contribution $-\int 4\epsilon R$ from I^- . These two contributions cancel each other, and therefore no conformal anomaly is seen at this level. In the usual AdS case, the conformal anomaly arises because the second boundary at I^- is absent, and one assumes that the metric is smooth in the interior and yields no further divergent contributions. In the degenerating cases with $\Lambda < 0$ that were described above, the precise region of integration for the coordinate is quite complicated: the integration over t runs from $+\infty$ to the point where the metric degenerates, which is given by (51). The Liouville action should arise in that case due to this peculiar region of integration of t , but we have not checked this; see [48] for a related calculation.

In any case, for de Sitter we find that the on-shell action is zero, since the conformal anomalies on I^- and I^+ cancel each other. Therefore, the right interpretation of this zero should be that it represents the difference of the Polyakov action for two-dimensional gravity on the two boundaries,

$$S \sim \Gamma_{I^+}(\hat{g}^+) - \Gamma_{I^-}(\hat{g}^-) \quad (55)$$

For solutions of the equation of motion S vanishes, just as it did in the case of free scalar fields in the S-matrix picture once we identified the in and out operators. However, once we view the two boundaries as independent, (55) represents the two Polyakov actions on each boundary, that only cancel each other once we identify the boundaries and take a classical solution of the equations of motion. Thus (55) seems to be the natural effective action from the S-matrix point of view, and it would be interesting to verify it directly using an S-matrix type calculation.

2.3.1 Further properties of the general solution

By looking at (50), we can analyze when and whether the metric degenerates at some point in the interior. This does not necessarily mean that the space is singular, it is also possible that we need another set of coordinates to describe the global structure of the solution. Nevertheless, it is interesting to see when this happens. If we take the boundary of dS_3 to have the topology of a two-sphere, then the scalar curvature is negative, and there is no holomorphic quadratic differential. Therefore, we see from (50) that the metric degenerates only if for some point on S^2

$$R \pm |T^\phi| \geq 0. \quad (56)$$

For a pure two-sphere which corresponds to global de Sitter space, we took $\phi = -2\log(1 + w\bar{w})$, and it follows that $R = -2/(1 + w\bar{w})^2$ and $T^\phi = 0$. Therefore, the left hand side of (56) is always negative and the metric does not degenerate, as expected. A small deformation of the metric on the two-sphere is still allowed, as long as the corrections decay sufficiently fast as $w, \bar{w} \rightarrow \infty$ so that they do not dominate over $R = -2/(1 + w\bar{w})^2$ for large w, \bar{w} . As the perturbations become larger, they reach a point where the metric degenerates somewhere in the bulk.

It is also interesting to compute the Brown-York stress tensor for the general solution. Using the prescription of [6] we find that the Brown-York stress-tensor at I^+ is given by

$$T_{ij}^{BY} = -\frac{1}{8\pi} \begin{pmatrix} -2T^\phi & \frac{1}{2}R \\ \frac{1}{2}R & -2\bar{T}^\phi \end{pmatrix}. \quad (57)$$

We see that the Brown-York stress tensor contains the Liouville stress tensor T^ϕ . Although we normally use the Brown-York stress tensor to define conserved quantities, if the boundary is a generic deformation of the two-sphere there are no isometries of the boundary geometry and there are no obvious conserved quantities associated to this Brown-York tensor. Nevertheless, it seems to know about the masses of e.g. de Sitter like solutions with point like defects. These can be constructed by taking

$$\phi = \log \left(\frac{\gamma^2 w^{\gamma-1} \bar{w}^{\gamma-1}}{(1 + w^\gamma \bar{w}^\gamma)^2} \right) \quad (58)$$

which corresponds to taking a two-sphere from which a piece has been cut out with the boundaries identified, so that conical singularities appear at the north and south poles. This Liouville field has

$$R = -2 \frac{\gamma^2 w^{\gamma-1} \bar{w}^{\gamma-1}}{(1 + w^\gamma \bar{w}^\gamma)^2} + \pi(\gamma - 1)\delta^2(w, \bar{w}) \quad (59)$$

and

$$T = \frac{-1 + \gamma^2}{2w^2}. \quad (60)$$

The curvature R indeed has a singularity at $w = \bar{w} = 0$, and the L_0 component of T is closely related to the mass of the corresponding conical defect spaces as computed in [6].

It would be interesting to generalize the above observations, study their connection with the mass conjecture of [6], and the relation with the results of [53].

2.3.2 On-shell action and entropy

We now discuss the connection between the on-shell action and the entropy of dS space; according to [54], the entropy is related the dimension of the Hilbert space of the dual theory, although this is not the point of view that we will take. In

the Euclidean case, the identification of the entropy with the on-shell value of the action seems to be natural from the Gibbons-Hawking Euclidean approach [55]. Note that the euclidean on-shell action as computed above yields precisely the Bekenstein-Hawking entropy [56]. If we take time imaginary with period $2\pi l$ [55, 57], and insert this into (54), we see that the the exponentials of time cancel each other, and the action becomes $-(l/4G) \int d^2w R$. With the conventions used here, the integral of R is -2π , and the action becomes $S = \pi l/2G$.

2.4 The Chern-Simons perspective

In this section we revisit the case of pure gravity in $2+1$ dimensions from the Chern-Simons perspective. In some sense, the duality between the $2+1$ -dimensional Chern-Simons theory and the $1+1$ WZW model is a prototype example of an AdS/CFT correspondence, and understanding the Chern-Simons theory for de Sitter space may lead to a better understanding of the nature of the dS/CFT correspondence.

In particular, we will study the nature of the boundary conditions that we have to impose on the Chern-Simons gauge fields. The relevant Chern-Simons theory has gauge group $SL_2(C)$. It turns out that the boundary conditions depend on the coordinate system and corresponding patch of de Sitter space we are considering. In the inflationary patch, the situation is very similar to the analogous situation discussed in the AdS/CFT correspondence [35, 36, 37]. The global patch is different, due to the two boundaries we have to consider. The final conclusion we will reach is similar to the conclusion reached in the previous section; for solutions of the equation of motion, the on-shell action is essentially trivial. Again, the difference with the AdS case lies in the assumption that the solution is regular in the interior. This is why the derivation of the Liouville action from Chern-Simons theory recently given in [58] does not apply to de Sitter space. We will also briefly discuss the gauge fields relevant for Kerr-de Sitter space.

The $SL_2(C)$ CS theory has been studied by Witten in [59]. It has level $t = k + is$, with k integer (and level $\bar{t} = k - is$ for the complex conjugate gauge field.) The usual $\Lambda > 0$ $2+1$ Einstein action arises when $k = 0$ and s is real. The action is real if s is real. The theory is also unitary for purely imaginary s , in which case there is a different inner product and the theory is related to Euclidean gravity with $\Lambda < 0$. We will come back to this observation in section 3. Explicitly, the action of gravity in $2+1$ dimensions with $\Lambda > 0$ is given by

$$S = \frac{is}{4\pi} (I_{CS}(A) - I_{CS}(\bar{A})), \quad (61)$$

with

$$I_{CS}(A) = \int d^3x \text{Tr}(A \wedge dA + \frac{2}{3} A^3) \quad (62)$$

the usual Chern-Simons functional. The gauge field A takes values in $SL(2, C)$, and \bar{A} is the complex conjugate of A . We have not yet included boundary contributions,

and the first step will be to determine a reasonable set of boundary conditions. The vielbein and spin-connections are given by

$$e = (A - \bar{A})/(2i), \quad \omega = (A + \bar{A})/2. \quad (63)$$

The following discussion is similar to the discussion for AdS space given in [58], but some of the details will be different. A first important remark is that for solutions of the equations of motion, the time-dependence of the gauge field can be encoded in a gauge transformation with

$$U = \begin{pmatrix} \cosh(t/2) & -i \sinh(t/2) \\ i \sinh(t/2) & \cosh(t/2) \end{pmatrix} \quad (64)$$

so that

$$A = U^{-1} \begin{pmatrix} \alpha^3/2 & \alpha^+ \\ \alpha^- & -\alpha^3/2 \end{pmatrix} U + U^{-1} \partial_t U \quad (65)$$

with α^3, α^\pm time independent. Actually, it is convenient to work with a different basis than the standard one. Introduce

$$N^\pm = \frac{1}{2}(\alpha^+ + \alpha^- \mp i\alpha^3), \quad N^3 = \frac{1}{2}(\alpha^+ - \alpha^-), \quad (66)$$

and the following basis for $SL(2, C)$,

$$T^+ = \begin{pmatrix} -i/2 & 1/2 \\ 1/2 & i/2 \end{pmatrix}, \quad T^- = \begin{pmatrix} i/2 & 1/2 \\ 1/2 & -i/2 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}. \quad (67)$$

This basis obeys the usual commutation relations, $[T^+, T^-] = 2T^3$ and $[T^3, T^\pm] = \pm T^\pm$. The connection (65) becomes

$$A = (dt + 2iN^3)T^3 + e^t N^+ T^- + e^{-t} N^- T^+ = U^{-1} \tilde{A} U + U^{-1} \partial_t U \quad (68)$$

with

$$\tilde{A} = 2iN^3 T^3 + N^+ T^- + N^- T^+. \quad (69)$$

The vielbein on spatial slices as $t \rightarrow \infty$ is therefore

$$e^{+\infty} = \lim_{t \rightarrow +\infty} e^{-t} \frac{A - \bar{A}}{2i} = \frac{1}{2i} (N^+ T^- - \bar{N}^+ T^+), \quad (70)$$

and similarly

$$e^{-\infty} = \lim_{t \rightarrow -\infty} e^t \frac{A - \bar{A}}{2i} = \frac{1}{2i} (N^- T^+ - \bar{N}^- T^-). \quad (71)$$

The metrics at $t \rightarrow \pm\infty$ are obtained from $\text{Tr}(ee)$ and we find $\frac{1}{2}N^+ \bar{N}^+$ respectively $\frac{1}{2}N^- \bar{N}^-$.

2.4.1 Global patch

To obtain the most general three dimensional solution we discussed in section 2.3 we take

$$N^+ = e^{\phi/2} dw, \quad N^- = \frac{1}{2} e^{-\phi/2} ((T^\phi - T(w)) dw + R d\bar{w}). \quad (72)$$

In addition, A needs to be flat, so that equivalently

$$2iN^3 T^3 + N^+ T^- + N^- T^+ \quad (73)$$

needs to be a flat connection. This implies that A is pure gauge if $\pi_1(M) = 0$. If $\pi_1(M) \neq 0$ we need to include holonomies around the non-contractible cycles in space-time. This may be relevant if we want to do the black hole and conical defect cases, where the topology is $S^1 \times R^2$ instead of $S^2 \times R$.

Flatness implies

$$\begin{aligned} dN^3 - iN^- \wedge N^+ &= 0 \\ dN^+ - 2iN^3 \wedge N^+ &= 0 \\ dN^- + 2iN^3 \wedge N^- &= 0. \end{aligned} \quad (74)$$

These are solved by

$$N^3 = \frac{1}{4i} (\bar{\partial}\phi d\bar{w} - \partial\phi dw). \quad (75)$$

In particular, for global de Sitter space we had $\phi = -2 \log(1 + w\bar{w})$, and the flat gauge field \tilde{A} reads

$$\tilde{A} = \begin{pmatrix} \frac{i}{2} & \frac{1-i\bar{w}}{2} \\ \frac{1+i\bar{w}}{2} & -\frac{i}{2} \end{pmatrix} \frac{dw}{1+w\bar{w}} + \begin{pmatrix} \frac{i}{2} & \frac{-1+iw}{2} \\ \frac{-1-iw}{2} & -\frac{i}{2} \end{pmatrix} \frac{d\bar{w}}{1+w\bar{w}}. \quad (76)$$

These gauge fields determine the behavior of the connection A as $t \rightarrow \pm\infty$, and therefore are intimately related to the boundary conditions we want to impose on A . The form of A_w does not appear immediately related to the form we need to perform a Hamiltonian reduction from an $SL_2(C)$ WZW model to a Liouville-like theory. The Sugawara stress tensor of such a WZW model living on the boundary would be proportional to $\text{tr}(A_w A_w)$, which in this case is equal to $\bar{w}^2/2(1+w\bar{w})^2 = \frac{1}{8} \partial\phi \partial\bar{\phi}$. The nonvanishing of this stress tensor can be viewed as evidence that de Sitter space in global coordinates should not be identified with the vacuum state in some conformal field theory, in agreement with the picture we develop later in this paper.

2.4.2 Inflationary patch

It is straightforward to obtain \tilde{A} and A in the inflationary patch, using (72) and (75), as the inflationary patch corresponds to $\phi = 0$. In particular, for \tilde{A} we find

$$\tilde{A} = T^- dw = \frac{1}{2} \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}. \quad (77)$$

Since T^- is conjugate in $SL_2(C)$ to the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (78)$$

this is of form needed to perform a Hamiltonian reduction from an $SL_2(C)$ WZW model to a Liouville-like theory, similar to what happens in de AdS case [60]. This can be further illustrated by looking at the form of \tilde{A} under a Virasoro transformation. In other words, we transform \tilde{A} with the Virasoro vector field

$$L_\xi = -\xi \partial_w + \frac{1}{2} \xi' \partial_t - \frac{1}{2} e^{-2t} \xi'' \partial_{\bar{w}} + \text{c.c.} \quad (79)$$

where $\xi(w)$ is holomorphic. Since gauge fields transform as one-forms, and $\tilde{A}_t = \tilde{A}_{\bar{w}} = 0$, \tilde{A} transforms into

$$\tilde{A} \rightarrow (1 - \xi') T^- dw. \quad (80)$$

An infinitesimal gauge transformation with parameter $\epsilon = \xi' T^3 + \xi'' T^+$ changes this, to first order in ξ , into

$$\tilde{A}_w \rightarrow (1 - \xi') T^- + [T^-, \epsilon] + \partial_w \epsilon = T^- + \frac{1}{2} \xi''' T^+. \quad (81)$$

Since the transform of the stress-tensor contains a term ξ''' , this transformed \tilde{A} is conjugate to a matrix of the form

$$\begin{pmatrix} 0 & cT \\ 1 & 0 \end{pmatrix} \quad (82)$$

with c some constant and T the stress tensor. Indeed, this is exactly of the form one needs to perform Hamiltonian reduction of the $SL_2(C)$ WZW model to a theory of induced $1+1$ dimensional gravity on the boundary.

2.4.3 Kerr-de Sitter solution

We take the metric for the Kerr-de Sitter space [61, 62] discussed in our previous paper [6]

$$ds^2 = -\frac{(r^2 + r_-^2)(r_+^2 - r^2)}{r^2} dt^2 + \frac{r^2}{(r^2 + r_-^2)(r_+^2 - r^2)} dr^2 + r^2 \left(d\phi + \frac{r_+ r_-}{r^2} dt \right)^2 \quad (83)$$

Following [62] we define

$$r^2 = \sinh^2 \rho r_-^2 + \cosh^2 \rho r_+^2 \quad (84)$$

which yields the metric

$$ds^2 = -d\rho^2 + \cosh^2 \rho (r_- dt + r_+ d\phi)^2 + \sinh^2 \rho (r_+ dt - r_- d\phi)^2. \quad (85)$$

This is again of the general form given in section 2.3, with ρ playing the role of time. Thus, we can use a gauge transformation with U as in (64), with t replaced by ρ , to remove the ρ dependence from the $SL_2(C)$ gauge field that describes the Kerr-de Sitter metric. The result is

$$\tilde{A} = \begin{pmatrix} 0 & \frac{1}{2}(r_- - ir_+)(d\phi - idt) \\ \frac{1}{2}(r_- - ir_+)(d\phi - idt) & 0 \end{pmatrix}. \quad (86)$$

This is the equation that is relevant to get the L_0 and \bar{L}_0 eigenvalues for the Kerr de Sitter spacetimes. In the present setup, we again identify A with the currents of a WZW model, and $\text{tr}(AA)$ with the Sugawara stress tensor of that WZW model. Since $\text{tr}(AA) \sim (r_- - ir_+)^2$ we indeed recover (up to factors of i) the mass and angular momentum of the Kerr de Sitter black hole.

Therefore, this type of analysis in terms of gauge fields provides valuable information about a putative dual theory. For the gauge field story of the BTZ black hole [63] consult [58].

2.4.4 On-shell action

Let us try to compute the value of the on-shell action in the Chern-Simons setup. This requires us first of all to determine the appropriate boundary terms. Following [58], we will impose boundary conditions that preserve the asymptotic form of the metric. As $t \rightarrow +\infty$, we therefore want to keep the form of N^+ fixed, which implies

$$\delta \text{Tr}(T^+ A) = 0, \quad t \rightarrow +\infty \quad (87)$$

and similarly as $t \rightarrow -\infty$, we want to fix N^- , so that

$$\delta \text{Tr}(T^- A) = 0, \quad t \rightarrow -\infty. \quad (88)$$

Also, in order to make sure there is no $dt dw$ or $dt d\bar{w}$ term in the metric, we also will impose

$$\text{Tr}(T^3(A - \bar{A})) = 0. \quad (89)$$

To examine the structure of the action, we decompose

$$A = A^+ T^- + A^- T^+ + A^3 T^3 \quad (90)$$

and therefore

$$\bar{A} = \bar{A}^+ T^+ + \bar{A}^- T^- - \bar{A}^3 T^3 \quad (91)$$

using the explicit form of T^i and the fact that \bar{A} is the complex conjugate of A .

Dropping the factor of $s/4\pi$, the variation of the CS action yields a boundary term

$$\delta S = -i \int_{\partial M} \text{Tr}(A^- \delta A^+ + A^+ \delta A^- + \frac{1}{2} A^3 \delta A^3) + \text{c.c.} \quad (92)$$

Now as $t \rightarrow \infty$, we want to find a boundary term proportional to δA^+ so that its vanishing implies the vanishing of the relevant boundary terms. This can be accomplished by adding

$$S_1 = i \int_{\partial M} \text{Tr}(A^+ \wedge A^-) + \text{c.c.} \quad (93)$$

To take care of (89), we add

$$S_2 = \frac{i}{2} \int_{\partial M} \text{Tr}(A^3 \wedge \bar{A}^3). \quad (94)$$

The total variation becomes

$$\delta(S + S_1 + S_2) = \int_{\partial M} \text{Tr}(-2iA^- \delta A^+ + 2i\bar{A}^- \delta \bar{A}^+ - \frac{i}{2}(A^3 + \bar{A}^3)\delta(A^3 - \bar{A}^3)) \quad (95)$$

which is indeed consistent with the boundary conditions as $t \rightarrow +\infty$.

On the other hand, on the other boundary at $t \rightarrow -\infty$, we want to put $\delta A^- = 0$, and this requires us to take $S - S_1 + S_2$ instead of $S + S_1 + S_2$. Notice that in all this discussion ∂M carries the standard orientation induced from that of M and the outward normal at each boundary.

The bulk action, for a solution of the equation of motion, equals

$$S = i \int_M \text{Tr}(A^3 A^+ A^- - \bar{A}^3 \bar{A}^+ \bar{A}^-) \quad (96)$$

where we used flatness of A to write the CS action as $-\frac{1}{3} \int A^3$. The components of A are given by

$$\begin{aligned} A^3 &= dt + 2iN^3 \\ A^+ &= e^t N^+ \\ A^- &= e^{-t} N^-. \end{aligned} \quad (97)$$

The total value of the on-shell action is

$$\begin{aligned} S &= i \int_M \text{Tr}(A^3 A^+ A^- - \bar{A}^3 \bar{A}^+ \bar{A}^-) \\ &\quad \pm i \int_{\partial M} \text{Tr}(A^+ \wedge A^- - \bar{A}^+ \wedge \bar{A}^-) \\ &\quad + \frac{i}{2} \int_{\partial M} \text{Tr}(A^3 \wedge \bar{A}^3). \end{aligned} \quad (98)$$

This seems to give (almost) zero. Notice that $A^+ \wedge A^-$ is proportional to the curvature, and the middle term gives something proportional to the curvature. The bulk term can be integrated and gives a logarithmically divergent term proportional to the curvature R . This is subtracted upon regularization, or replaced by a constant times the curvature. The last boundary term vanishes because it is proportional to

$N^3 \wedge N^3$ which vanishes (the field ϕ is real). (Note that the results here are a little different from those obtained earlier using the Einstein action for gravity since we are treating the boundary terms differently and are not being careful about finite terms left over from removing the logarithmic bulk divergence.)

The situation is completely different from AdS, where a similar calculation gives the Liouville action. This is among other things due to the fact that the boundary terms here are different from the ones in AdS [58]. Still, taking boundary terms similar to the ones in [58] does not improve the situation very much. The main difference is that de Sitter space has two boundaries, whereas AdS has a single boundary. The Liouville action arises in [58] precisely because one makes an assumption about the regularity of the solution in the interior, so that the only boundary contributions in partial integration is from the AdS boundary, although the coordinates one uses are not only bounded by the boundary of AdS but also by the region in the interior where the metric in those coordinates degenerates, as given by (50). It may be possible to similarly recover the Liouville action on the inflationary patch of de Sitter space using an assumption about the regularity at the horizon of the inflationary patch, but we have not verified this; see e.g. [53] for a construction of this type.

The situation is similar to what we discussed in section 2.3. The Liouville action does not arise from a naive calculation of the on-shell action. However, treating the the two boundaries as separate surfaces and carefully removing the bulk divergence in (98) by adding suitable boundary terms we would expect to find that the total action vanishes, but consists of two separate Liouville pieces on different boundaries that cancel.

Altogether the main lesson from this section is that it is rather subtle to impose the right boundary conditions, but for any choice of boundary condition the naive analysis of the bulk action as a function of boundary data yields an uninteresting result. However, by examining the contributions from the the two boundaries of de Sitter space separately we can get more interesting answers that agree with S-matrix type calculations in the bulk. This structure is evidence that if there is a dual theory it should be two (possibly entangled) CFT's living on separate spaces rather than one. In the next section we will argue that these CFT's are of a novel form, with different hermiticity conditions from what we are used to in conformal field theory. In section 4 we will then combine all these observations to arrive at a proposal of what a dual description of de Sitter space could look like.

3 Hermiticity and the inner product

If there is a CFT dual to de Sitter space there are also indications that it will have unconventional hermiticity and unitarity properties. For example, conventional unitarity suggests a upper bound on particle masses in de Sitter space [4] in an analog

of the Breitenlohner-Freedman bound [41] in AdS. Such a bound is difficult to understand from a physical point of view. However, there is reason to believe that a unitary theory with $SL(2, C)$ symmetry is relevant for three dimensional de Sitter space.

First of all, in the Chern-Simons formulation of 3d gravity, the $SL(2, C)$ CS theory [59] with level $t = k + is$, with k integer (and level $\bar{t} = k - is$ for the complex conjugate gauge field) describes both de Sitter space and Euclidean AdS. The usual $\Lambda > 0$ 2+1 Einstein action arises when $k = 0$ and s is real. The action is real if s is real. The theory is also found to be unitary for purely imaginary s , in which case there is a different inner product and the theory is related to Euclidean gravity with $\Lambda < 0$. Therefore, since Euclidean AdS is related to a Euclidean CFT with conventional inner product and hermiticity conditions, we expect some unusual features in the inner product for a dual to de Sitter.

A guess for the unusual inner product follows from the relation between CS theory in the bulk and WZW theory on the boundary. The CS action relevant for $\Lambda > 0$ is given in (61). On a manifold with boundary, the CS action is related to a WZW theory living on the boundary [64, 65]. The CS action for a compact gauge group reduces to a chiral WZW theory on the boundary. With gauge group $SL(2, C)$, one would therefore expect a chiral $SL(2, C)$ WZW theory on the boundary. However, there are two terms in the action (61). The first term gives rise to a chiral $SL(2, C)$ WZW theory, the second one to an antichiral WZW theory ([59], see also [53]). However, the gauge field \bar{A} was the complex conjugate of A , and therefore the antichiral WZW theory is related by complex conjugation to the chiral WZW theory. This is quite distinct from the case with $\Lambda < 0$, where A and \bar{A} are independent $SL(2, R)$ gauge fields. On the boundary we obtain independent chiral and antichiral $SL(2, R)$ WZW theories that combine into one standard $SL(2, R)$ theory.

In the $\Lambda > 0$ case that is relevant for de Sitter space, we seem to find a WZW-like theory on the boundary, but one with different hermiticity conditions. Namely, the hermiticity conditions also involve an exchange of left and right movers. One may check that on the level of zero modes, such hermiticity conditions are compatible with unitary representations of $SL(2, C)$. In fact, this is in perfect agreement with what we find for a free scalar field in de Sitter space; as we will elaborate in the next section, the Hilbert space of a free massive scalar field in de Sitter space consists of unitary representations of $SL(2, C)$.

An $SL(2, C)$ WZW theory was also used in [66] in an attempt to explain the entropy for de Sitter space a la Carlip [67]. That $SL(2, C)$ theory did however not live on the boundary of de Sitter space and is not obviously related to our discussion here.

A further motivation for these novel hermiticity conditions follows from computations of L_0 and \bar{L}_0 for various solutions of the Einstein equations with $\Lambda > 0$. Given the asymptotic 2d euclidean conformal symmetries of the dS_3 , the boundary stress tensor formalism can be used to compute the eigenvalues of L_0 and \bar{L}_0 for a

variety of conical defect and spinning defect (Kerr-dS) spacetimes [6]. These eigenvalues are not consistent with a unitary representation of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ as we might have expected. Rather, there are consistent with the (unitary) principal series representations of $SL(2, \mathbb{C})$.

So far, attempts to find de Sitter solutions in string theory frequently involve unusual reality condition as for example in the work of Hull [1]. There, de Sitter always seems to arise in cases where reality conditions are unusual and wrong sign fields appear³. These wrong signs may be natural from the point of view of a unitary theory with $SL(2, \mathbb{C})$ symmetry. As we will speculate in the conclusions, this might all makes sense in a new class of string theories, where the hermiticity conditions are not the usual ones.

With all of these points in mind we will now explore a simple toy model of the unusual Hermiticity conditions for 2d CFT suggested by the Chern-Simons analysis of de Sitter space described above.

3.1 A toy model

We arrive at our toy model by considering the simplest Chern-Simons theory, that of $U(1)$. On the boundary of a three-manifold, it gives rise to a chiral boson. However, in case of dS_3 , we have a complex $SL(2, \mathbb{C})$ gauge field and its complex conjugate, and a certain reality condition relating the two. On the boundary, this becomes two chiral WZW theories, each with $SL(2, \mathbb{C})$ current algebra. The Euclidean AdS reality condition tells us that both chiral currents should be viewed as $SL(2, \mathbb{R})$ currents, rather than $SL(2, \mathbb{C})$ currents, and subsequently the two chiral $SL(2, \mathbb{R})$ models can be combined in a single $SL(2, \mathbb{R})$ WZW model. The other reality condition, relevant for dS_3 , is a suitable relation between the two currents, roughly of the form

$$J_L^\dagger = J_R \quad (99)$$

where the subscripts L, R denote the left and right moving sector.

In our toy model, we consider $U(1)$ currents with the reality condition

$$J_L^\dagger = i J_R \quad (100)$$

which leads to the Hermiticity conditions on the Virasoro generators that we expect to be appropriate for the de Sitter case. Thus, we consider a complex left-moving chiral boson ϕ and a complex right moving chiral boson χ . The currents are

$$J_L = i\partial\phi = \sum \alpha_n z^{-n-1}, \quad J_R = i\bar{\partial}\chi = \sum \beta_n \bar{z}^{-n-1} \quad (101)$$

³Another problem with this set-up is the fact that it is hard to talk about de Sitter gravity in lower dimensions; solutions found in [1] always involve de Sitter space times a non-compact manifold.

and we will assume the usual commutation relations between α_n and β_n ,

$$[\alpha_m, \alpha_n] = [\beta_m, \beta_n] = m\delta_{m+n,0}, \quad (102)$$

but not the usual hermiticity conditions. The hermiticity conditions that follow from (100) suggest to take

$$\alpha_n^\dagger = i\beta_n, \quad \beta_n^\dagger = i\alpha_n. \quad (103)$$

A different but equivalent choice of hermiticity conditions is to take $\alpha_n^\dagger = \beta_{-n}$, and $\beta_n^\dagger = \alpha_{-n}$. This shows that the new hermiticity conditions are obtained by combining the usual ones with an exchange of left and right movers.

The Virasoro generators should be a normal-ordered version of the usual answer $T = -\frac{1}{2}\partial\phi\partial\phi$. The normal ordering depends on a choice of vacuum etc. We will not discuss this issue yet, but simply note that with a suitable normal ordering the hermiticity conditions become⁴

$$L_k^\dagger = -\bar{L}_k \quad (104)$$

which then also implies that the central charges should obey

$$c_L^* = -c_R \quad (105)$$

which is compatible with a purely imaginary central charge for both the left and right movers. (This type of relation was encountered in our discussion of the mass of de Sitter space in [6].) In the following we will refer to a Virasoro algebra with these Hermiticity conditions as a “Euclidean Virasoro algebra” since it arose from considering possible Euclidean CFTs dual to de Sitter space.

The hermiticity conditions (104) give rise to unitary representations of $SL(2, C)$. In general, given some Lie algebra and some anti-linear operator \dagger that squares to one, the algebra that will be unitarily implemented is the -1 eigenspace of \dagger . One then readily verifies that the combinations $L_m + \bar{L}_m$ and $i(L_m - \bar{L}_m)$ with eigenvalue -1 under the hermiticity conditions (104) form the $SL(2, C)$ Lie algebra, and therefore they are unitarily implemented.⁵

3.1.1 Representations

Stone-von Neumann representations: To study the representations of the oscillator algebra (103) we construct the hermitian combinations

$$\delta_n = \alpha_n + i\beta_n, \quad \epsilon_n = i(\alpha_n - i\beta_n). \quad (106)$$

⁴An equivalent set of hermiticity conditions would be to take $L_k^\dagger = \bar{L}_{-k}$ instead.

⁵It is worth remembering why the usual hermiticity condition $L_m^\dagger = L_{-m}$ gives $SL(2, R)$ rather than $SU(2)$ as unitarily represented subalgebra. Clearly, $L_k^\dagger = -L_k$ gives $SL(2, R)$, but the other one seems like the hermiticity condition for $SU(2)$. This point is a bit subtle, but the solution is that in terms of Pauli matrices, $i\sigma_1, i\sigma_2, i\sigma_3$ generate $SU(2)$, whereas $\sigma_1, \sigma_2, i\sigma_3$ generate $SL(2, R)$, in a somewhat nonstandard basis. Given $L_m^\dagger = L_{-m}$, the (-1) subalgebra is generated by $i\sigma_3 \sim iL_0, \sigma_1 \sim L_1 - L_{-1}, \sigma_2 \sim i(L_1 + L_{-1})$ and therefore gives a unitary representation of $SL(2, R)$ rather than $SU(2)$.

The commutation relations between these objects read

$$[\delta_m, \delta_n] = [\epsilon_m, \epsilon_n] = 0, \quad [\delta_m, \epsilon_n] = 2im\delta_{m+n,0}. \quad (107)$$

This is exactly the same as the usual relations between coordinates and momenta, and the representations are known from the Stone-von-Neumann theorem. The unique irreducible unitary representation of the Hilbert space is

$$\otimes_{n>0} L^2(\delta_n) \otimes |p\rangle. \quad (108)$$

where $L^2(\delta_n, \epsilon_n)$ are the L^2 normalizable functions of δ_n and ϵ_n . Here, $|p\rangle$ is a representation of the zero modes that satisfies

$$(\alpha_0 + i\beta_0)|p\rangle = p_1, \quad i(\alpha_0 - i\beta_0)|p\rangle = p_2 \quad (109)$$

for real p_1, p_2 . Thus, $\alpha_0|p\rangle = (p_1 - ip_2)/2|p\rangle$ and $\beta_0|p\rangle = (p_2 - ip_1)/2|p\rangle$.

Actually, it is not quite clear that the Stone-von Neumann representation is also the unique one in case we have an infinite number of degrees of freedom, but we have not found any other obvious representations of the commutations relations.

Highest weight representations: We could also try to build more conventional representations by imposing the following conditions

$$\alpha_n|p\rangle = \beta_{-n}|p\rangle, \quad n > 0 \quad (110)$$

on a highest weight state $|p\rangle$ that obeys (109). Though interesting, it is not clear that these states can be organized in a unitary representation. To define an inner product we need a bilinear form

$$B(\prod_{n_i} \beta_{n_i} \prod_{m_j} \alpha_{-m_j} |p\rangle, \prod_{n_k} \beta_{n_k} \prod_{m_l} \alpha_{-m_l} |p\rangle). \quad (111)$$

There are several possibilities. First, one can try to define an inner product compatible with the hermiticity conditions. But then $B(\alpha_{-1}|p\rangle, \alpha_{-1}|p\rangle) = B(|p\rangle, i\beta_{-1}\alpha_{-1}|p\rangle) = 0$ and the inner product is not positive definite. One can also define a more conventional inner product on the highest weight modules, as we do for the usual free field. This yields a positive definite inner product, but now the oscillators do not obey the required hermiticity conditions. Later we will comment on the relation between these representations and the Hilbert space of a free scalar field in de Sitter space.

3.1.2 Calculations in the toy model

Calculations in this toy model are not completely straightforward. One first needs to construct a state in the Hilbert space (108). The state $|p\rangle$ is by itself not an element of the Hilbert space, because 1 is not a square integrable function. One can of course

write down an infinite set of square integrable functions of the variables δ_n , and use the corresponding state to do calculations. In particular, one can compute correlation functions of the field $\delta\phi$ in such a state. A particularly convenient basis for $L^2(\delta_n)$ is to use harmonic oscillator wave functions. These are of the form $(\delta_n + i\epsilon_{-n})^k |\psi_n\rangle$, where $|\psi_n\rangle$ is a state annihilated by the annihilation operator $\delta_n - i\epsilon_{-n}$. These states are not invariant under $SL(2, C)$, so we cannot use symmetry principles to determine the form of two and three point functions as in usual conformal field theory. It would be interesting to study the nature of these correlations functions in more detail.

3.1.3 Virasoro representations

We return to the representation (108) which has a positive definite norm and respects the hermiticity conditions. It is interesting to decompose this into representations of the Virasoro algebra, with the new hermiticity conditions $L_k^\dagger = -\bar{L}_k$. The representation theory of this Virasoro algebra should be a suitable generalization of the representation theory of $SL(2, C)$.

It is convenient to rewrite the Hilbert space slightly using complex coordinates $z_n = (\delta_n - i\epsilon_n)/\sqrt{n}$. (Then z_n and \bar{z}_n are rescaled versions of α_n and β_n .) The Hilbert space is the set of square integrable functions of z_n, \bar{z}_n , times the highest weight state $|p\rangle$. For the Virasoro generators we find $L_m \sim \sum_n z_n \partial/\partial z_{m-n}$ and similarly for their complex conjugates. This is reminiscent of the form of the Virasoro generators in the old matrix model [68]; perhaps there is an interesting connection here.

So what are the representations of the Euclidean Virasoro algebra with hermiticity conditions (104)? By analogy with the usual case a reasonable guess could be that for generic eigenvalues of the zero modes, the oscillator Hilbert space furnishes an irreducible unitary representation of the Euclidean Virasoro algebra. This is also reasonable because $SL(2, C)$ has its unitary representations on square integrable functions of a single complex variable (with standard measure for the principal representations). If the modes $\{L_{\pm 1}, L_0, \bar{L}_{\pm 1}, \bar{L}_0\}$ have such a representation, it is quite reasonable that adding more modes of the Virasoro algebra will require extra sets of square integrable functions of a complex variable. It would be interesting to develop this representation theory in more detail.

In terms of the complex functions, and with a suitable normal ordering that we still have not specified, the lowest order Virasoro generators will become⁶

$$\begin{aligned} L_{-1} &= - \sum_{n>0} \sqrt{n(n+1)} z_n \frac{\partial}{\partial z_{n+1}} - \tilde{\alpha}_0 \frac{\partial}{\partial z_1} \\ L_0 &= \frac{1}{2} \tilde{\alpha}_0^2 - \sum_{n>0} n z_n \frac{\partial}{\partial z_n} \end{aligned}$$

⁶We employ $\alpha_n = z_n \sqrt{n}$, $\alpha_{-n} = -\sqrt{n} \partial/\partial z_n$, $\beta_n = -i\sqrt{n} \bar{z}_n$, $\beta_{-n} = -i\sqrt{n} \partial/\partial \bar{z}_n$ for $n > 0$.

$$L_1 = - \sum_{n>0} \sqrt{n(n+1)} z_{n+1} \frac{\partial}{\partial z_n} + \tilde{\alpha}_0 z_1 \quad (112)$$

where $\tilde{\alpha}_0 = (p_1 - ip_2)/2$ and we have left out a normal ordering constant in L_0 . Similar expressions can be written down for the antiholomorphic generators. To be compatible with the Hermiticity conditions, the normal ordering constant is chosen as

$$L_0 = \frac{1}{2} \alpha_0^2 - \sum_{n>0} n(z_n \frac{\partial}{\partial z_n} + \frac{1}{2}) \quad (113)$$

(Although this expression looks infinite, we will see that this expression that also happens to have a well-defined spectrum.) This is also compatible with the commutation relations of the Virasoro algebra since there is an ambiguity that appears when we evaluate the commutator $[L_1, L_{-1}]$. In it appears the sum

$$\sum_{n>0} (z_{n+1} \partial_{n+1} - z_n \partial_n) \sqrt{n(n+1)} \quad (114)$$

that can be evaluated to give $-\sum_{n>0} n z_n \partial_n$, but also to give $-\sum_{n>0} n \partial_n z_n$. The choice of normal ordering given above amounts to picking the answer “in the middle” of these two expressions.

The other positive Virasoro generators are

$$L_m = - \sum_{n>0} \sqrt{(m+n)n} z_{m+n} \frac{\partial}{\partial z_n} + \alpha_0 \sqrt{m} z_m + \frac{1}{2} \sum_{n=1}^{m-1} \sqrt{n(m-n)} z_n z_{m-n} \quad (115)$$

and similar expressions for the negative Virasoro generators. We conjecture that with these differential operators there is a unitary irreducible representation of the Euclidean Virasoro algebra on the space of complex square integrable functions of z_k, \bar{z}_k .

3.1.4 Intermezzo: A non-positive definite realization of the hermiticity conditions

Before studying unitary representations of the Euclidean Virasoro algebra, we first provide some details of a non-positive definite realization. Using unitary representations of $SL(2, C)$ is rather complicated. As above, we can stay closer to the usual intuition regarding highest and lowest weight states, and sacrifice unitarity.

Define a pairing $B(|\psi\rangle, |\chi\rangle)$, which is antilinear in the first and linear in the second argument, in such a way that

$$B(|\psi\rangle, L_k |\chi\rangle) = B(-\bar{L}_k |\psi\rangle, |\chi\rangle) \quad (116)$$

and similarly for \bar{L}_k . We give up unitarity by not demanding positive definiteness of this bilinear form. The natural Hilbert space on which such a pairing acts is

generated by a highest weight state $|h, \bar{h}\rangle$ which is annihilated by L_k , $k > 0$ and \bar{L}_{-k} , $k > 0$. Notice the similarity to the discussion in [59]. The pairing can be written in conventional form as follows. Given a state $\prod_{i,j} L_{-k_i} \bar{L}_{l_j} |h, \bar{h}\rangle$ we define a bra state

$$\langle -\bar{h}^*, -h^* | \left(\prod_{i,j} L_{-k_i} \bar{L}_{l_j} \right)^\dagger \quad (117)$$

where the dagger operation is our new hermiticity condition, and h^* is the complex conjugate of h . The two entries in the highest weight state denote its L_0 and \bar{L}_0 eigenvalue.

The inner product of two states is now described as usual by the product of a bra and a ket state. The latter is completely fixed once we use the Virasoro algebra, and use that

$$\langle -\bar{h}^*, -h^* | h, \bar{h} \rangle = 1. \quad (118)$$

This defines a non-positive definite inner product, with respect to which we do have the new hermiticity conditions. What are the properties of this inner product? Many states have norm zero. However, there is a modified notion of positive definiteness. Consider the \mathbf{Z}_2 involution that sends L_k to $-\bar{L}_{-k}$ and similarly for \bar{L}_k . This is not an anti-involution, like \dagger , but a linear map, and it gives us a \mathbf{Z}_2 involution of the Hilbert space we have defined. Call this involution ω . Then we have the property

$$B(\omega|\psi\rangle, |\psi\rangle) = B(|\psi\rangle, \omega|\psi\rangle) \geq 0. \quad (119)$$

Therefore, the $+1$ eigenspace of ω has positive norm, the -1 eigenspace has negative norm. This suggests to project out the -1 eigenspace of ω . However, in this new Hilbert space we no longer have a representation of the full Virasoro algebra, but only of the generators that commute with ω , which are $L_k - \bar{L}_{-k}$. The \mathbf{Z}_2 invariant Hilbert space carries therefore a unitary representation of $SL(2, R)$, but not of $SL(2, C)$.

This situation recalls the quantization of a free scalar field in de Sitter space, where solutions to the free wave equation do not have positive definite Klein-Gordon norm. Also, the map ω is reminiscent of the CPT map discussed in [3].

A place where the above discussion could become especially relevant is in elliptical de Sitter space, discussed first by Schrödinger [69]. Elliptical de Sitter space is the quotient of de Sitter space by the antipodal map, which maps $X_i \rightarrow -X_i$ in the definition (5) of de Sitter space, resulting in a manifold that is not time orientable. (Also see [70].) This \mathbf{Z}_2 action is exactly like the map ω , and the $+1$ eigenspace of ω in the above setup is an interesting candidate for more conventional CFT description of elliptical de Sitter space.

3.1.5 Spectrum of L_0, \bar{L}_0

The zero momentum contributions to L_0 and \bar{L}_0 read $(p_1 - ip_2)^2/8$ and $(p_2 - ip_1)^2/8$, which satisfy $L_0^\dagger = -\bar{L}_0$, as required by hermiticity. To determine the complete

spectrum of L_0 and \bar{L}_0 is not so easy. Consider the subset of the Hilbert space spanned by the square integrable functions of z_1, \bar{z}_1 . On these functions, L_0 acts as $-z_1\partial_1$, and \bar{L}_0 acts as $-\bar{z}_1\bar{\partial}_1$. Below we will argue that these operators act on wavefunctions just like generators of $SL(2, C)$ acting on a principal series representation. We will use this to determine the spectrum of our free scalar with unusual Hermiticity conditions and, in the next section, use this to explore the partition function.

First, we briefly review the principal series representations of $SL(2, C)$. These are realized on functions of a complex variable $f(z)$, and $SL(2, C)$ acts as

$$T_g f(z) = \frac{1}{(cz + d)^{\nu_1}(\bar{c}\bar{z} + \bar{d})^{\nu_2}} f\left(\frac{az + b}{cz + d}\right). \quad (120)$$

The inner product is given by

$$\langle f, g \rangle = \int d^2 z f(\bar{z}) g(z). \quad (121)$$

Strictly speaking, the action of $SL(2, C)$ is defined on test functions on the complex plane that vanish faster than any power at infinity. At the end, we take the Hilbert space completion of the set of test functions. One may check that the inner product is invariant only if

$$\nu_1 + \bar{\nu}_2 - 2 = 0. \quad (122)$$

In addition, the function is only well-defined if $\nu_1 - \nu_2 \in \mathbf{Z}$.

The case with $\nu_1 = \nu_2 = 1 + 2i\gamma$, $\gamma \in \mathbf{R}$ corresponds to the usual principal series representations (see [71, 44, 43]). The simplest case is the one with $\gamma = 0$, so that $\nu_1 = \nu_2 = 1$. Taking $a = 1 + \epsilon$ and $d = 1 - \epsilon$ and expanding to first order in ϵ , we get

$$f(z) \rightarrow 2\epsilon\left(\frac{\nu_1}{2} + z\partial\right)f(z) + 2\bar{\epsilon}\left(\frac{\nu_2}{2} + \bar{z}\bar{\partial}\right)f(z). \quad (123)$$

Therefore, in the principal series representation with $\nu_1 = \nu_2 = 1$ the scaling generators in $SL(2, C)$ act as $\mathcal{L}_0 = -(z\partial + 1/2)$ and $\bar{\mathcal{L}}_0 = -(\bar{z}\bar{\partial} + 1/2)$, exactly matching each term in the series in (113) including the $1/2$. Therefore, it appears that we can compute the spectrum of L_0 for our free boson with novel Hermiticity conditions by first computing the spectrum of \mathcal{L}_0 and $\bar{\mathcal{L}}_0$ and then summing over principal series representations.

An eigenfunction of \mathcal{L}_0 with eigenvalue λ , and with eigenvalue $\bar{\lambda}$ of $\bar{\mathcal{L}}_0$, must be $z^{-\lambda-1/2}\bar{z}^{-\bar{\lambda}-1/2}$. This function fails to be integrable at infinity, zero, or both. Nevertheless, the operators \mathcal{L}_0 and $\bar{\mathcal{L}}_0$ have a well-defined spectrum⁷ in such a principal series representation [71]. The spectrum is such that $(\mathcal{L}_0 - \bar{\mathcal{L}}_0) \in \mathbf{Z}$ and $i(\mathcal{L}_0 + \bar{\mathcal{L}}_0) \in \mathbf{R}$. The first of these conditions is natural, since the first operator corresponds to rotations in two dimensions and therefore generates a $U(1)$.

⁷A number λ is in the spectrum of an operator A if $(A - \lambda)^{-1}$ exists, is bounded, and is defined on a dense set.

Armed with this information, and looking back at (113) we see that eigenfunctions of L_0, \bar{L}_0 can be taken as a product of functions of the z_i , each of which is an eigenfunction of $\mathcal{L}_0, \bar{\mathcal{L}}_0$ in a principal series representation. Altogether, we find that the total spectrum of L_0, \bar{L}_0 is precisely as it is in a principal series representation.

3.1.6 Characters of Euclidean Virasoro

The partition function of our Euclidean Virasoro algebra is closely related to the character formula for $SL(2, C)$. In [72] the definition of the character of a unitary representation of locally compact groups is reviewed. The result for $SL(2, C)$ is described, for example, in [73]. Take $\nu_1 = 1 - m/2 - i\rho/2$, $\nu_2 = 1 + m/2 - i\rho/2$. Then the character reads

$$\text{Tr} R \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \frac{|\lambda|^{i\rho-m}\lambda^m + |\lambda|^{-i\rho+m}\lambda^{-m}}{|\lambda - \lambda^{-1}|^2}. \quad (124)$$

Up to signs this is somewhat reminiscent of the usual character formula for $SU(2)$. If the representation has $\nu_1 = \nu_2 = 1$, we get simply

$$\text{Tr} R \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \frac{2}{|\lambda - \lambda^{-1}|^2}. \quad (125)$$

Now the matrix appearing in the left hand side is simply the unitary operation $U = e^{-2 \log \lambda \mathcal{L}_0 - 2 \log \bar{\lambda} \bar{\mathcal{L}}_0}$ and we get

$$\text{Tr}(q^{\mathcal{L}_0} \bar{q}^{\bar{\mathcal{L}}_0}) = \frac{2}{(q^{1/2} - q^{-1/2})(\bar{q}^{1/2} - \bar{q}^{-1/2})}. \quad (126)$$

This is real, which is consistent with the fact that the operator appearing here is unitary. Recall that the L_0 and \bar{L}_0 operators appearing in (113) look like an infinite sum of \mathcal{L}_0 and $\bar{\mathcal{L}}_0$ operators in principal series representations of $SL(2, C)$. Now, using the hermitian operators $X = i(L_0 + \bar{L}_0)$ and $Y = L_0 - \bar{L}_0$ we extrapolate to find the Virasoro character

$$\text{Tr}(q_1^X q_2^Y) = \sum_{\alpha_0, \beta_0} q_1^{i(\alpha_0^2 + \beta_0^2)/2} q_2^{(\alpha_0^2 - \beta_0^2)/2} \prod_{k=1}^{\infty} \frac{2}{((q_1^i q_2)^{k/2} - (q_1^i q_2)^{-k/2})((q_1^i q_2^{-1})^{k/2} - (q_1^i q_2^{-1})^{-k/2})}. \quad (127)$$

From this interesting equation for the Euclidean Virasoro character two natural questions arise: (a) Can we compute the density of states from this expression?, (b) Can we build modular invariants from it? Actually our expression for the Virasoro character (127) is somewhat formal. For instance, it clearly vanishes if $|q_1^i q_2| \neq 1$ because the terms in the product vanish for large k . We therefore view (127) as a formal expression that encodes part of the structure of the Hilbert space. Notice that in more conventional variables the above expression is tantalizingly close to the

ordinary Virasoro character

$$\mathrm{Tr}(q^{L_0} \bar{q}^{\bar{L}_0}) = q^{\alpha_0^2/2} \bar{q}^{\beta_0^2/2} \prod_{k=1}^{\infty} \frac{2}{(q^{k/2} - q^{-k/2})(\bar{q}^{k/2} - \bar{q}^{-k/2})}, \quad (128)$$

but we have formally analytically continued to values where q and \bar{q} are no longer each other's complex conjugate.

3.1.7 The interpretation of these novel CFT's

We have gone a long way towards implementing the novel reality conditions suggested by de Sitter physics in a toy model of a free scalar. There are some interesting features, such as an unbounded spectrum of L_0 , but perhaps that is what we should expect for a time-dependent background.

To what extend are these theories ordinary 1 + 1-dimensional field theories? A field theory in 1 + 1 dimensions has as local symmetry group a semidirect product of translations in the σ, t directions and $SO(1, 1)$ Lorentz rotations. The hermitian generators are $K = i(t\partial_\sigma + \sigma\partial_t)$, $P_t = i\partial_t$ and $P_\sigma = i\partial_\sigma$. These form a simple algebra, the only nonzero commutators are $[P_t, K] = iP_\sigma$ and $[P_\sigma, K] = iP_t$. Can this be a subalgebra of $sl(2, C)$? Let's consider the basis $X_k = i(L_k + \bar{L}_k i)$, $Y_k = L_k - \bar{L}_{-k}$ of hermitian generators. The commutators are

$$[X_k, X_l] = -[Y_k, Y_l] = i(k - l)X_{k+l}, \quad [X_k, Y_l] = [Y_k, X_l] = i(k - l)Y_{k+l}. \quad (129)$$

We find that $P_t = X_{-1}$, $P_\sigma = Y_{-1}$ and $K = -X_0$ satisfy the right algebra. Therefore, there is an extension of the symmetry group in 1 + 1 Lorentzian dimensions which is $SL(2, C)$.

The novel CFT's that we consider here can therefore be ordinary 1 + 1 dimensional field theories, where the Poincaré group is extended to $SL(2, C)$, rather than $SL(2, R) \times SL(2, R)$. This is not the same as a Euclidean conformal field theory, which carries unitary representations of $SL(2, R) \times SL(2, R)$.

One may try to construct a Lagrangian for the toy model by starting with Lagrangians for chiral bosons, and combining these in a suitable way. It would be interesting to work this out in more detail.

3.2 Application to de Sitter Holography

Above we have implemented the novel reality conditions suggested by de Sitter physics in a toy model of a free scalar. We now explore how CFTs realizing our Euclidean Virasoro algebra would be expected to represent the physics of de Sitter space.

As we have seen in earlier sections, there are many reasons to believe that two CFT's, rather than one, are relevant. De Sitter has two boundaries rather than one (although they are not completely independent), and in the static patch we see a

thermal spectrum which typically arises in situations where one entangled factor in a product of two field theories is integrated out. Indeed, the thermal state of free fields in the static patch can be understood in terms of integrating out the correlated region in the antipodal patch. The argument that a dual to 3d de Sitter would involve only one CFT [4] is partly based on the fact that in global coordinates there is an $SL(2, C)$ invariant vacuum [39], which is what we have in a single CFT⁸. Likewise, the isometry group of 3d de Sitter is a single $SL(2, C)$ which again suggests one dual CFT, not two.

However, there is a potential way around this dilemma. It is possible that de Sitter is described by a pure, but entangled state in two CFTs which preserves a single $SL(2, C)$ and that tracing over a suitable part of the Hilbert space yields a thermal density matrix.⁹ In conventional tensor products of CFT's, the only $SL(2, C)$ invariant state is the product of the vacua of the two theories. The tensor product of two highest weight states is not invariant under $1 \otimes L_0 + L_0 \otimes 1$.

Once we change the hermiticity conditions, the situation changes completely. We can quite generally construct $SL(2, C)$ invariant states, using the unitary representations of section 3.1.1, but also using the unitary but not positive definite realization of section 3.1.4.

Below we discuss these possibilities for realizing an $SL(2, C)$ invariant state in a product of two CFTs associated with the early and late time de Sitter boundaries. Next, we give a concrete example of an $SL(2, C)$ -invariant, pure state in the tensor product of two of our toy models. We then study how quite generally the Hilbert space could be split up in terms of Hilbert spaces associated with antipodal static patches, realizing a kind of de Sitter complementarity. We conclude by discussing the picture of de Sitter duality emerging from our explorations.

3.2.1 Pure states with $SL(2, C)$ Invariance

Using the novel hermiticity conditions: Consider the unitary $SL(2, C)$ representation with $\nu_1 = \nu_2 = 1$. As discussed above, this representation appears in the toy model, and it can be extended to a representation of the Virasoro algebra with the new hermiticity conditions. If we call this representation C , then $C \otimes C$ contains the identity representation, which is crucially different from the case where we take two discrete highest weight representations of $SL(2, R)$. We can see this since there is a map from $C \otimes C$ to the complex numbers (which form the trivial representation of $SL(2, C)$):

$$(\phi_1(z), \phi_2(z')) \longrightarrow \int d^2z \phi_1(z) \phi_2(z). \quad (130)$$

⁸In the toy model, however, there is no obvious $SL(2, C)$ invariant ‘vacuum state’.

⁹This would be reminiscent of the situation in the eternal black BTZ black hole [37, 74].

We also could have chosen

$$(\phi_1(z), \phi_2(z')) \longrightarrow \int \frac{d^2 z d^2 z'}{|z - z'|^2} \phi_1(z) \phi_2(z'). \quad (131)$$

One can check that both these expressions are invariant under the action of $SL(2, C)$. The second one will not play any major role here but would be important if we were not using the representation with $\nu_1 = \nu_2 = 1$. Using this pairing, we can make a pure state in the tensor product of two Euclidean Virasoro representations, that is invariant under $SL(2, C)$. We simply take the dual map of (130) or (131), and consider the image of an arbitrary nonzero complex number under the dual map.

An explicit example of such a pure, entangled state will be given in the following subsection.

Non-unitary pure states with $SL(2, C)$ invariance We can also describe the pure states that appear when we use as Hilbert space a combination of highest and lowest weight representations. The construction proceeds exactly as above, but we do not use the fact the tensor product of two principal series representations contains the identity. Instead we observe that the tensor product of a highest and a lowest weight representation with highest and lowest weight $+h$ and $-h$ respectively, contains the identity representation. It is slightly cumbersome to describe this pure state. First, we introduce another \mathbf{Z}_2 involution, called I , that sends $L_k \rightarrow -L_{-k}$ and similarly for \bar{L}_k . In contrast to the dagger operation, this is a linear, not an antilinear operation. We also need the standard inner product which we know from string theory, and denote that by a pairing $B_{\text{standard}}(|\psi\rangle, |\chi\rangle)$. Let $|\psi_\alpha\rangle$ be a basis for the module built on $|h, \bar{h}\rangle$. Then $I|\psi_\alpha\rangle$ is a basis for the module built on $|-h, -\bar{h}\rangle$. We also define $M_{\alpha\beta} = B_{\text{standard}}(|\psi\rangle_\alpha, |\psi\rangle_\beta)$, and its inverse $M^{\alpha\beta}$. Then the pure state is

$$\sum_{\alpha, \beta} M^{\alpha\beta} |\psi_\alpha\rangle \otimes I|\psi_\beta\rangle \quad (132)$$

and one easily verifies it preserves the diagonal Virasoro subalgebra with $L_m, m \geq -1$, and $\bar{L}_m, m \leq +1$. While this construction gives an $SL(2, C)$ invariant state, it does not arise from a unitary implementation of $SL(2, C)$. Again, this state may be relevant for a dual description of the Z_2 quotient of de Sitter space by the antipodal map, as discussed in section 3.1.4.

3.3 An example of a pure entangled state

To describe an example of a pure entangled state with $SL(2, C)$ invariance, we take two copies of the toy model. The modes of the two models are denoted by $\alpha_n^1, \beta_n^1, \alpha_n^2, \beta_n^2$. We introduce the following two linear combinations

$$U_n = \frac{1}{2}(\alpha_n^1 + i\beta_{-n}^1 + i\alpha_n^2 - \beta_{-n}^2)$$

$$V_n = \frac{1}{2}(\alpha_n^1 - i\beta_{-n}^1 + i\alpha_n^2 + \beta_{-n}^2). \quad (133)$$

The commutations relations between these operators are

$$[U_n, U_m^\dagger] = -[V_n, V_m^\dagger] = n\delta_{n-m,0}. \quad (134)$$

We will call U, V annihilation operators and U^\dagger, V^\dagger creation operators. The nice feature of these operators is that $L_k^1 + L_k^2$ and $\bar{L}_k^1 + \bar{L}_k^2$ will only consist of terms that contain one creation operator and one annihilation operator, but no terms with two creation or two annihilation operators. To show this, we take a typical term $\alpha_k^1 \alpha_l^1 + \alpha_k^2 \alpha_l^2$ in $L_{k+l}^1 + L_{k+l}^2$. Using

$$\begin{aligned} \alpha_n^1 &= \frac{1}{2}(U_n + U_{-n}^\dagger + V_n - V_{-n}^\dagger) \\ \alpha_n^2 &= \frac{1}{2}(-iU_n + iU_{-n}^\dagger - iV_n - iV_{-n}^\dagger) \end{aligned} \quad (135)$$

we get

$$2(\alpha_k^1 \alpha_l^1 + \alpha_k^2 \alpha_l^2) = (U_k + V_k)(U_{-l}^\dagger - V_{-l}^\dagger) + (U_{-k}^\dagger - V_{-k}^\dagger)(U_l + V_l) \quad (136)$$

and

$$2(\beta_k^1 \beta_l^1 + \beta_k^2 \beta_l^2) = (U_k - V_k)(U_{-l}^\dagger + V_{-l}^\dagger) + (U_{-k}^\dagger + V_{-k}^\dagger)(U_l - V_l) \quad (137)$$

Because the diagonal $SL(2, C)$, spanned by $L_k^1 + L_k^2$ and $\bar{L}_k^1 + \bar{L}_k^2$ with $k = -1, 0, 1$, contains only terms linear in creation and annihilation operators, an entangled $SL(2, C)$ invariant state is the $|0\rangle >$ that satisfies

$$U_n|0\rangle = V_n|0\rangle = 0. \quad (138)$$

This way of constructing an $SL(2, C)$ invariant state is quite reminiscent of the construction of such states in the Hilbert space of a free scalar field on de Sitter space. In that case, the $SL(2, C)$ generators are also sums of products of creation and annihilation operators, as we will explain in more detail in section 4.

If we consider excitations of the state $|0\rangle$, we find excitations with negative norm. This is because the true annihilation operators are $U_n, U_{-n}^\dagger, V_{-n}, V_n^\dagger$ with $n \geq 0$. Therefore, $|0\rangle$ is not a really good candidate $SL(2, C)$ invariant state. We can do better by starting with a state $|\Omega\rangle$ which is annihilated by the true annihilation operators. Then we can construct a squeezed state

$$|\text{entangled}\rangle = \exp \left[\sum_{k>0} \frac{1}{k} (U_k^\dagger V_k + U_{-k} V_{-k}^\dagger) \right] |\Omega\rangle \quad (139)$$

which is $SL(2, C)$ invariant, as one may verify.

If one tries to construct an $SL(2, C)$ invariant state for a single copy of the toy model, one finds that similar tricks do not work. This we view as an encouraging

sign towards the correctness of the holographic dual picture of de Sitter space we are proposing here. It is also worth pointing out that the above construction bears in many ways a close similarity to the construction of boundary states. This analogy may be useful when trying to find $SL(2, C)$ invariant states in more complicated examples.

With this explicit example of an $SL(2, C)$ invariant state in hand, it is in principle possible to start doing explicit calculations of correlations functions in the product of two toy models, and to compare those with bulk calculations; we leave this to future work, as well as a detailed analysis of the normal ordering issues that appear in this construction.

3.3.1 Static versus global patch and de Sitter thermodynamics

It is a well-known fact that the static observer in de Sitter space sees a thermal spectrum of excitations when de Sitter is placed in the Euclidean vacuum state. One way of understanding this thermality for a scalar field in the static patch is by integrating out the fluctuations in the antipodal static patch. The static patch of de Sitter space also has a horizon giving rise to gravitational entropy. Integrating out quantum fields in the antipodal patch cannot quite account for this entropy because of the usual ultraviolet problems afflicting such “geometric entropies”. Instead, in previous explorations holography, 3d de Sitter entropy was associated with the degeneracy of particular states in a highest weight representation of a dual CFT. For the Kerr-de Sitter spaces with “mass” M and angular momentum J , it was found that these states would have $L_0 \sim M + iJ$ and similarly for \bar{L}_0 , where L_0, \bar{L}_0 are generators of time translation in the static patch but are radial translations in the plane at the late time de Sitter boundary. Applying the Cardy formula naively to such states yielded the de Sitter entropy [6, 7, 62, 58].

Here we explore how the physics of antipodal static patches could be implemented in a holographic setting where a dual to de Sitter involves two CFTs, one associated with each of the early and late time boundaries. Denote the Hilbert spaces of these two theories as $\mathcal{H}_{i,f}$. We assume that both $\mathcal{H}_{i,f}$ are CFT’s based on the Euclidean Virasoro algebra, since $SL(2, C)$ acts separately at plus and minus infinity. Since global de Sitter space has a single $SL(2, C)$ isometry group, we expect it to be represented in the dual theory as an entangled state with an $SL(2, C)$ invariance, perhaps constructed as described above.

How can we construct Hilbert spaces $\mathcal{H}_{L,R}$ associated with two antipodal static patches from $\mathcal{H}_{i,f}$? From the action of $SL(2, C)$ on the bulk of de Sitter space, we know that neither of $\mathcal{H}_{L,R}$ should carry an action of $SL(2, C)$ by itself. It is interesting to learn how this mixing works. For that purpose, consider a $U(1)$ that rotates by an angle θ along the equator of the sphere at equal time slices in global de Sitter. Since

the isometry group is really $SL(2, C)/\mathbf{Z}_2$, the rotation is implemented by

$$U(\theta) = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix} \quad (140)$$

On the 2-sphere boundary of de Sitter at early or late times, this rotation is the map $z \rightarrow -1/z$, which is quite similar to the CPT map of Witten [3]. One can easily see how it acts on e.g. L_0 , by computing $U(\theta)L_0U(\theta)^{-1}$. One gets

$$U(\theta)L_0U(\theta)^{-1} = \cos(2\theta)L_0 - \sin(2\theta)(L_{-1} + L_1). \quad (141)$$

Thus, for a rotation through π , L_0 flips signs.

Recall from above that the holographic discussions of de Sitter entropy [6, 7, 62, 58] suggest that e.g. \mathcal{H}_L is built out of standard highest weight modules. (It was using this assumption that the entropy of the static patch was reproduced from a CFT dual to de Sitter.) The fact that L_0 flips sign in going between two antipodal static patches then naturally means that \mathcal{H}_R is built out of lowest weight modules. To get some sense of the relation between the Hilbert spaces $\mathcal{H}_{i,f}$ based on the Euclidean Virasoro algebra and $\mathcal{H}_{L,R}$ we should compare the partition function in the former case with the partition function in a highest/lowest weight representation. In the absence of a concrete proposal for a CFT dual to de Sitter space we again explore the free scalar field from Sec. 3.1. The trace of $q^{L_0}\bar{q}^{\bar{L}_0}$ in $\mathcal{H}_{i,f}$ is given in (128) while the trace in highest/lowest weight representations is well known:

$$\begin{aligned} \chi_i = \chi_f &\sim \prod_{k>0} \frac{1}{|q^{k/2} - q^{-k/2}|^2} \\ \chi_L &\sim \prod_{k>0} \frac{1}{|1 - q^k|^2} \\ \chi_R &\sim \prod_{k>0} \frac{1}{|1 - q^{-k}|^2}, \end{aligned} \quad (142)$$

where χ denotes the trace of $q^{L_0}\bar{q}^{\bar{L}_0}$ over the relevant Hilbert space (with the naive inner products). (We have left out numerical factors and zero modes for the moment.)

Interestingly,

$$\chi_i\chi_f = \chi_R\chi_L \quad (143)$$

which in turn suggests that

$$\mathcal{H}_f \otimes \mathcal{H}_i \equiv \mathcal{H}_L \otimes \mathcal{H}_R. \quad (144)$$

This suggests that there are two different ways of thinking about the same Hilbert space: we could either regard it as the Hilbert space of two CFTs defined at past and future infinity of de Sitter space realizing Euclidean Virasoro algebras, or as the

Hilbert space of antipodal static patches based on lowest/highest weight representations¹⁰. The two radically different bases would have a complicated, possibly non-local relationship to each other, perhaps realizing a sort of horizon complementarity [76] in de Sitter space. We will explore this further that the level of free fields in de Sitter space in Sec. 4.

3.3.2 A picture of holographic duality in de Sitter

In summary, our picture is that the $SL(2, C)$ invariant vacua of de Sitter space should be realized as entangled states in $\mathcal{H}_L \otimes \mathcal{H}_R$, with the thermally entangled state corresponding to the Euclidean vacuum. This entanglement would account for de Sitter when entropy when one of the factors is traced over. Other entanglements would give rise to the one-parameter family of de Sitter vacua.¹¹ All of these entangled states should translate into $SL(2, C)$ -invariant states in $\mathcal{H}_i \otimes \mathcal{H}_f$, on which the Euclidean Virasoro symmetry acts in a simple way. It would be very interesting to firmly establish that such a picture, including the one-parameter family of vacua, is realized at least in a toy model.

This picture does not contradict the picture of de Sitter space given by Witten in [3]. In that paper, the claim is that all quantum gravity can do is to provide an Hilbert space structure for de Sitter space. A connection with our proposal can be made, once we identify the set of all correlation functions of one of the two Euclidean CFT's with some large vector space (in the spirit of the Chern-Simons - conformal field theory relation). The Hilbert space structure then arises from the entangled state. Given two correlation functions in each of the CFT's, the Hilbert space structure is then given by computing the joint correlation function in the entangled state that links the two CFT's.

4 Further evidence: free scalar field

A great deal was learned about the AdS/CFT correspondence by studying the physics of free scalar fields in AdS space. Therefore, in this section we study the massive, free scalar field in de Sitter, in order to collect further evidence for the picture of dS holography that we have given above. Free, massive scalars in de Sitter space have been extensively studied in the literature (see, e.g, [39], and the recent works [7, 40]).

¹⁰Incidentally, yet another point of view is that the rotation above does not really map a state into a state, but rather it maps a particle into an antiparticle. In that case, one should view \mathcal{H}_L as the bra states, and \mathcal{H}_R as the ket states, and the $U(1)$ rotation mixes bra and ket states. This is again somewhat similar to [3]. 't Hooft [75] has also discussed a similar idea in the context of black holes; in this case bra and ket states were associated to different regions.

¹¹It is possible that once interactions are taken into account all vacua except the Euclidean one will be eliminated.

Here, we will focus on the representations that the Hilbert space of a free scalar field form under the action of the isometry group. The representation theory corroborates the picture of de Sitter holography developed above. Along the way we will recover many known results and also find several new ones.

First, we will review why isometries are implemented unitarily on the Hilbert space of a canonically quantized scalar field. Next, we study the Hilbert space of the scalar field, and show that the one-particle states built from an $SO(d, 1)$ -invariant vacuum state form a principal series representation of $SO(d, 1)$ (sometimes called class 1 representations). Next, we determine what form states have that have a fixed eigenvalue under $L_0 + \bar{L}_0$ (or its generalization to $SO(d, 1)$). When restricted to the static patch, these are states with a fixed frequency ω . We also find a basis of states in which the decomposition of the Hilbert space in modes living on the two separate static patches becomes particularly simple. This latter decomposition also provides a clue regarding which states we should trace over in a holographic dual description, in order to find the density matrix that describes one static patch. Finally, we briefly consider the particularly simple case of a massless scalar in $d = 2$.

4.1 Isometries

First we reexamine the way isometries of a space are implemented on the Hilbert space of a canonically quantized scalar field. Consider a space with metric (for simplicity)

$$ds^2 = -N^2 dt^2 + g_{ij} dx^i dx^j. \quad (145)$$

If this space has an isometry $\delta t = \lambda, \delta x^i = \xi^i$, we have the usual equations

$$\begin{aligned} \lambda \partial_t N + \xi^i \partial_i N + N \partial_t \lambda &= 0, \\ -N^2 \partial_i \lambda + g_{ij} \partial_t \xi^j &= 0, \\ \lambda \partial_t g_{ij} + \xi^k \partial_k g_{ij} + \partial_i \xi^k g_{kj} + \partial_j \xi^k g_{ki} &= 0. \end{aligned} \quad (146)$$

The field equation of a free scalar reads

$$\partial_t (N^{-1} \sqrt{g} \partial_t) \phi - \partial_i (N \sqrt{g} g^{ij} \partial_j) \phi + m^2 N \sqrt{g} \phi = 0, \quad (147)$$

with a Klein-Gordon norm, $\langle \phi, \chi \rangle = -i \int dx \sqrt{g} N^{-1} (\phi \dot{\chi}^* - \dot{\phi} \chi^*)$, which is time-independent ($\partial_t \langle \phi, \chi \rangle = 0$), and invariant under the isometries ($\langle \delta \phi, \chi \rangle + \langle \phi, \delta \chi \rangle = 0$). To prove the latter statement, one can do an explicit calculation, using the equations of motion of the scalar field and the various equations for an isometry. The fact that isometries leave the Klein-Gordon norm invariant suggests that isometries give rise to anti-hermitian operators in the quantum theory, so that the isometry group is unitarily implemented. To verify this, we have to understand in more detail the relation between the Klein-Gordon norm and canonical quantization.

To do canonical quantization, we first write the most general solution of the equations of motion

$$\phi = \sum a_m u_m(x, t) \quad (148)$$

where u_m includes all modes, including positive and negative frequency modes. In general, u_m^* is also a set of solutions, and can therefore be expressed in terms of u_m ,

$$u_m^* = \sum_n H_{mn} u_n. \quad (149)$$

The canonical momentum $\pi = \sqrt{g} N^{-1} \partial_t \phi$ satisfies $[\phi(t, x), \pi(t, x')] = i\delta(x - x')$, and in addition the field and momentum commute with themselves. Inserting the expansion (148) we find that

$$\begin{aligned} \sum_{m,n} [a_m, a_n] \sqrt{g}(x', t) N^{-1}(x', t) u_m(x, t) \dot{u}_n(x', t) &= i\delta(x - x') \\ \sum_{m,n} [a_m, a_n] u_m(x, t) u_n(x', t) &= 0 \\ \sum_{m,n} \sqrt{g}(x, t) N^{-1}(x, t) \sqrt{g}(x', t) N^{-1}(x', t) [a_m, a_n] \dot{u}_m(x, t) \dot{u}_n(x, t) &= 0. \end{aligned} \quad (150)$$

We can integrate these identities against u_r and \dot{u}_r to get various identities involving the object

$$L_{mn} = \int dx \sqrt{g} N^{-1} u_m \dot{u}_n \quad (151)$$

in terms of which the commutator is very simple

$$[a_m, a_n] = -i(L_{mn} - L_{nm})^{-1}. \quad (152)$$

Thus, the commutator is expressed directly in terms of the Klein-Gordon norm

$$L_{mn} - L_{nm} = i\langle u_m, u_n^* \rangle_{KG} \quad (153)$$

and together with the hermiticity conditions in (149) that imply

$$\sum_m a_m^\dagger H_{mn} = a_n \quad (154)$$

we see that all canonical structure is encoded in the Klein-Gordon norm. To canonically quantize a field we seek a representation of the commutation relations plus hermiticity conditions. The Klein-Gordon norm may or may not be the inner product on the resulting Hilbert space.

The fact that isometries preserved the KG norm implies that they preserve the canonical commutation relations. In some sense, this result should have been expected. We already knew that isometries, via Noether theorem, give rise to a conserved charge that becomes an operator. They are thus canonical transformations, generated by commuting with the appropriate conserved charge. Canonical transformations always preserve the commutation relations.

What about unitarity? Since isometries are real, they commute with the operation of complex conjugation. Therefore, if Q is the operator that implements the isometry, then $[Q, a^\dagger] = [Q, a]^\dagger$. This shows that Q is anti-hermitian. Hence, we have showed that isometries give rise to a canonical transformation, with time-independent coefficients, generated by an anti-hermitian operator. Therefore, any standard quantization of the commutation relations and the hermiticity conditions will automatically give a unitary representation of the isometry group.

Incidentally, in Witten's approach to de Sitter space [3] we take an inner product on the Hilbert space that is not compatible with the hermiticity condition in (149), but one that follows from the path integral. If the naive inner product was $\langle \cdot, \cdot \rangle_{\text{standard bulk}}$, then

$$\langle \phi, \chi \rangle_{\text{witten}} = \langle \omega(\phi), \chi \rangle_{\text{standard bulk}}. \quad (155)$$

Here, ω represents the antipodal map on the sphere. In other words, combining the antipodal map with the standard inner product in the bulk of de Sitter gives the usual Hermiticity conditions in a dual Euclidean CFT. Therefore, in Witten's inner product we find a unitary representation of $SO(2, d-1)$ rather than $SO(1, d)$. However, the thermal spectrum in [7] was computed using the standard quantization rather than Witten's inner product. This provides additional evidence for the $SL(2, C)$ structure that we are advocating. Another interesting perspective on Witten's inner product has been provided in [7].

4.2 The one-particle Hilbert space of a massive scalar

Before looking more closely at the massive scalar field, we first give some expressions for the $SO(d, 1)$ generators in terms of the global coordinates on de Sitter space. Recall that de Sitter space is described by the equations

$$-X_0^2 + X_1^2 + \dots + X_d^2 = 1 \quad (156)$$

in a space with metric with signature $(-, +, \dots, +)$. The entire space can be parametrized by

$$X_0 = y \sinh t, \quad X_i = y_i \cosh t \text{ for } i > 0 \quad (157)$$

with $y^2 = y_1^2 + \dots + y_d^2$ and de Sitter space is obtained by restricting to $y = 1$. Normally, we would choose spherical coordinates on the $d-1$ -sphere parametrized by y_i with $y = 1$, but here we find it more convenient to work with y_i instead, keeping in mind that $y = 1$.

The generators of $SO(d, 1)$ are given by

$$M_{ij} = X_i \frac{\partial}{\partial X_j} - X_j \frac{\partial}{\partial X_i}, \quad K_i = X_0 \frac{\partial}{\partial X_i} - X_i \frac{\partial}{\partial X_0}. \quad (158)$$

The generators M_{ij} are obviously the rotations in d dimensions, whereas K_i are boosts. When expressed in terms of the coordinates (157), the generators become

$$\begin{aligned} M_{ij} &= y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i} \\ K_i &= \frac{y_i}{y} \frac{\partial}{\partial t} + \tanh t \left(y \frac{\partial}{\partial y_i} - \frac{y_i}{y} \sum_k y_k \frac{\partial}{\partial y_k} \right). \end{aligned} \quad (159)$$

One may check that these generators leave y invariant, and therefore reduce to the generators of $SO(d, 1)$ acting on de Sitter space, once we replace y_i by spherical coordinates.

Consider now a massive scalar field on de Sitter space with mass m . A convenient basis of solutions of the equations of motion are for example the so-called Euclidean modes and their complex conjugates. An explicit expression can be found in eq. (3.36) in [7]. Euclidean modes are characterized by the fact that they admit a regular analytic continuation to the lower half of the d -sphere that represents Euclidean de Sitter space. The complex conjugates of the Euclidean modes are regular in the upper half-sphere. These properties are preserved by acting with isometries, and therefore the isometries map Euclidean modes into a linear combination of the Euclidean modes, not involving their complex conjugates. Therefore, the Euclidean modes furnish a representation of $SO(d, 1)$. What does this representation look like? The Euclidean modes are in one-to-one correspondence with elements of $L^2(S^{d-1})$. Indeed, the properly normalized Euclidean modes are of the form $f_k(t)Y_k(\Omega)$, where Y_k is a spherical harmonic for S^{d-1} .

We can therefore set up a one-to-one correspondence between $L^2(S^{d-1})$ and the Euclidean modes by assigning to $\sum c_k Y_k(\Omega)$ the Euclidean mode $\sum c_k f_k(t)Y_k(\omega)$. If the Euclidean modes are properly normalized, this correspondence maps the standard inner product on $L^2(S^{d-1})$ to the Klein-Gordon inner product of the Euclidean modes, so it really is an isomorphism of Hilbert spaces. We argued that the Euclidean modes formed a representation of $SO(d, 1)$, and we therefore also must find a unitary representation of $SO(d, 1)$ on $L^2(S^{d-1})$, once we use this correspondence.

What is the explicit form of the generators (159) when acting on $L^2(S^{d-1})$? The answer is that they become

$$\begin{aligned} M_{ij} &= y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i} \\ K_i &= -2h^+ \frac{y_i}{y} + \left(y \frac{\partial}{\partial y_i} - \frac{y_i}{y} \sum_k y_k \frac{\partial}{\partial y_k} \right). \end{aligned} \quad (160)$$

Here h^+ is the scaling dimension given in (30), $2h^+ = (d-1)/2 + i\mu$, with $\mu = \sqrt{m^2 - (d-1)^2/4}$. To prove (160), we first observe that by general arguments, the

generators M_{ij} will remain the same, and that K_i has to be of the form

$$c \frac{y_i}{y} + \left(y \frac{\partial}{\partial y_i} - \frac{y_i}{y} \sum_k y_k \frac{\partial}{\partial y_k} \right)$$

for some constant c . The constant c can be determined, for example, by acting on the Euclidean mode corresponding to the constant spherical harmonic. After some algebra, and using the recursion relation

$${}_2F_1(a, b, 2a, z) + (z-1) {}_2F_1(a+1, b+1, 2a+1, z) = \frac{2a-b}{2(2a+1)} z {}_2F_1(a+1, b+1, 2a+2, z) \quad (161)$$

one obtains $c = -2h^+$.

One may for instance compare (160) with $d = 3$ to the principal series representations given in (120). We find that (159) is an infinitesimal version of (120) with $\nu_1 = \nu_2 = 2h^+$. For general d , the generators (160) generate a principal series (or class one) representation of $SO(d, 1)$ on $L^2(S^{d-1})$. It is straightforward to check that this representation is unitary.

Similarly, the complex conjugates of the Euclidean modes transform under a principal series representation, with $2h^+$ replaced by $2h^-$ in (160). Since the Euclidean vacuum is annihilated by the operators that multiply the Euclidean modes, and the one particle states are obtained by acting with their hermitian conjugates, the one-particle states made from this vacuum transform exactly as the complex conjugate Euclidean modes. Therefore, the one-particle states form a principal series representation of $SL(2, C)$. This proves statements made in earlier sections, and is further evidence in favor of the new hermiticity conditions we have proposed.

Notice that $SL(2, C)$ maps annihilation operators into annihilation operators, and creation operators into creation operators. Therefore, the generators can be written as sums of products of creation and annihilation operators, and therefore the state annihilated by all annihilation operators (the Euclidean vacuum) is indeed $SL(2, C)$ invariant. This structure is nicely mimicked in our example of an $SL(2, C)$ invariant state in the product of two toy models in section 3.2.2.

To give an example how $SO(d, 1)$ acts on S^{d-1} , consider $d = 2$. Then $SO(2, 1) = SL(2, R)$, and $SL(2, R)$ acts on a circle via

$$e^{i\theta} \rightarrow \frac{e^{i(\alpha+\theta)} \cosh \xi + e^{i\beta} \sinh \xi}{e^{i(\theta-\beta)} \sinh \xi + e^{-i\alpha} \cosh \xi} \quad (162)$$

where ξ, α, β are real numbers parametrizing $SL(2, R)$.

The description of the Hilbert space in terms of functions on S^{d-1} is clearly in the right direction of holography, since all reference to the time dependence has been lost. Usually holographic descriptions lose spatial directions, but here time has been eliminated. This is in keeping with the proposal in [6] and [77] (also see [78]) that time evolution in a space with a positive cosmological constant is related to inverse RG flow in a dual Euclidean field theory.

4.3 Eigenfunctions of a boost

In the static patch, time translations are clearly an isometry. This isometry corresponds to a boost generator of $SO(d, 1)$. There are different choices of static patch, and we will restrict our attention to the static patch where time translation is given by the boost K_1 . Solutions of the equation of motion that in the static patch behave as $e^{i\omega t}$ therefore obey

$$K_1 f = i\omega f. \quad (163)$$

If we use the correspondence between Euclidean modes and $L^2(S^{d-1})$, corresponding to f there should be a function ψ_ω on the sphere, that has the property that

$$K_1 \psi_\omega = i\omega \psi_\omega. \quad (164)$$

One can readily solve this differential equation and one finds

$$\psi_\omega = \left(\frac{1 + y_1}{1 - y_1} \right)^{i\omega/2} (1 - y_1)^{-h^+} \chi(\Omega') \quad (165)$$

where $\chi(\Omega_{d-2})$ is some function on the unit $d - 2$ -sphere in the $y^2 \dots y^d$ directions; it can be represented by a homogeneous function of degree zero of y_2, \dots, y_d .

If one were to solve the laplace equation on the $d - 1$ -sphere, and would write the metric as $ds^2 = d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2$ (so that $y_1 = \cos \theta$), one also gets a decomposition of the form $h(y_1)\chi(\Omega_{d-2})$, but now $h(y_1)$ will typically be a Gegenbauer polynomial. Thus, the frequency eigenmodes (165) form a completely different basis compared to the usual spherical harmonics.

To summarize the meaning of (165), suppose we would decompose (165) in spherical harmonics $\sum c_k Y_k$ on S^{d-1} , and suppose we would write down the corresponding Euclidean mode $\sum c_k f_k(t) Y_k$. This Euclidean mode, when restricted to the static patch, will depend on static time t_s as $e^{it_s \omega}$. In particular, the Bogoliubov transformation that relates modes on the static patch to global modes can be extracted from the decomposition of (165) in spherical harmonics.

It is a straightforward exercise to show that

$$\int dy_1 dy_2 \dots dy_d \delta(y - 1) \psi_\omega(y_i) \bar{\psi}_{\omega'}(y_i) \sim \delta(\omega - \omega') \quad (166)$$

so that the boost eigenfunctions really behave as Fourier modes on the sphere. They are not really well-defined functions on the sphere, due to their behavior near $y = \pm 1$. This does not come as a surprise, since we already noticed before that L_0 -eigenfunctions did not exist in the toy model of section 3.

So far, we restricted attention to Euclidean modes. However, there is a one-parameter family of $SO(d, 1)$ invariant vacua, obtained by taking linear combinations of creation and annihilation operators. A priori, it may seem strange that this can be done, since the Euclidean modes transform as (160), and their complex conjugates

as (160) with h^+ replaced by h^- . How can a linear combination of Euclidean modes and their complex conjugates still transform nicely under $SO(d, 1)$? The point is that the representation (160) and the one with h^+ replaced by h^- are equivalent representations. In particular, if $g(y)$ transforms as (160) with h^+ replaced by h^- , then

$$\mathcal{T}(g)(y) \equiv \int d^d z \delta(z-1) (1 - \sum_k y_k z_k)^{-2h^+} g(z) \quad (167)$$

transforms precisely as in (160). The integration kernel of this transform is precisely like a CFT two-point function of two operators with weight h^+ . Therefore, one might expect to make a more precise translation between properties of functions on S^{d-1} and properties of a putative holographically dual CFT [4, 40, 7].

Two particular examples of different basis are to use the in and out modes discussed for example in section 2.2. These are related to the Euclidean modes in a simple way,

$$\begin{aligned} \phi^E &\sim \phi^{\text{in}} + e^{2\pi i h^+} (\phi^{\text{in}})^* \\ \phi^E &\sim \phi^{\text{out}} + e^{-2\pi i h^-} (\phi^{\text{out}})^*. \end{aligned} \quad (168)$$

In particular, in odd dimensions $e^{2\pi i(h^+ + h^-)} = 1$, and the in and out modes are the same. Interestingly, the coefficients that appear in (168) are similar to terms that appear in (165) if we try to analytically continue y_1 to values less than -1 or greater than $+1$. Perhaps all the different choices of modes and vacua in de Sitter space can be identified with (165) with different choices of analytic continuations over the real axis. This is not an unreasonable idea, as the behavior near $y_1 = \pm 1$ of ψ_ω is closely related to behavior of the solutions of the field equations near the cosmological horizon.

An important question is what the decomposition of $L^2(S^{d-1})$ in terms of modes living on the two static patches (the northern and southern diamonds) looks like. Naively, one might be inclined to simply cut the $d-1$ -sphere into two pieces, since this is what happens at the $t=0$ slice in global coordinates. However, this is too naive, as the map between functions on S^{d-1} and solutions of the equations of motion is more complicated than this. The map sent $\sum c_k Y_k$ to $\sum c_k f_k(t) Y_k$, and even at $t=0$ the values of f_k are highly non-trivial. Therefore, cutting up the sphere is not the right thing to do. It would be nice to have a detailed derivation of the right way to decompose things, but for now we merely present a guess. The guess is that the functions on S^{d-1} that correspond to modes on the northern diamond are precisely the eigenfunctions (165) with $\omega > 0$, whereas the ones on the southern diamond are the ones with $\omega < 0$. This would fit in nicely with our proposed picture of de Sitter complementarity in Sec. 3.3.1.

As a first check of this idea, one would like to see the thermal nature of the static patch appear. For this, we use a rather peculiar basis of functions on S^{d-1} , namely

$$h_k(y_i) = (1 - y_1)^{1/2 - h^-} (iy + \sqrt{1 - y^2})^k \chi(\omega_{d-2}) \quad (169)$$

for some integer k . If we integrate this basis function h_k against $\psi_\omega(y_i)$, we run into the integral

$$S_k(\omega) = \int_{-1}^1 dy_1 \frac{1}{\sqrt{1-y_1^2}} \left(\frac{1+y_1}{1-y_1} \right)^{i\omega/2} (iy + \sqrt{1-y^2})^k. \quad (170)$$

One can show, using contour deformation, that for $k \neq 0$

$$S_k(-\omega) = (-1)^{k-1} e^{\omega\pi} S_k(\omega). \quad (171)$$

The reason we had to introduce the peculiar basis (169) is that in this way we isolate the thermal factor $e^{\omega\pi}$ up to the phase $(-1)^{k-1}$. Up to some subtleties involving this minus sign, we indeed see that decomposition gives the required thermal behavior. This is evidence that the decomposition in the two static patches is simply the decomposition in states with $\omega > 0$ and $\omega < 0$.

We leave a more detailed study of this proposal and its application to the toy model given in Sec. 3 to future work.

4.4 Example: massless field in two dimensions

Finally, we illustrate some of the above observations by looking at a massless field in two dimensional de Sitter space. We begin by summarizing the coordinate systems on $-X_0^2 + X_1^2 + X_2^2 = 1$.

Global coordinates: These obey

$$X_0 = \tan \tau \quad ; \quad X_1 = \cos \phi / \cos \tau \quad ; \quad X_2 = \sin \phi / \cos \tau, \quad (172)$$

with metric

$$ds^2 = \frac{1}{\cos^2 \tau} (-d\tau^2 + d\phi^2). \quad (173)$$

The $SL(2, R)$ action is

$$\begin{aligned} L_0 &= -2 \cos \tau \cos \phi \partial_\tau + 2 \sin \phi \sin \tau \partial_\phi \\ L_1 + L_{-1} &= 2 \cos \tau \sin \phi \partial_\tau + 2 \cos \phi \sin \tau \partial_\phi \\ L_1 - L_{-1} &= 2 \partial_\phi. \end{aligned} \quad (174)$$

The Euclidean modes are the modes $\exp(im\phi - i|m|\tau)$, with m an integer. One immediately verifies that $SL(2, R)$ maps these modes into themselves, and that this gives a representation of the form (160) on $L^2(S^1)$.

Static coordinates: These obey

$$X_0 = -\sinh t / \cosh \rho \quad ; \quad X_1 = \cosh t / \cosh \rho \quad ; \quad X_2 = \tanh \rho, \quad (175)$$

with metric

$$ds^2 = \frac{1}{\cosh^2 \rho} (-dt^2 + d\rho^2). \quad (176)$$

The $SL(2, R)$ action is

$$\begin{aligned} L_0 &= 2\partial_t \\ L_1 + L_{-1} &= -2 \cosh t \sinh \rho \partial_t - 2 \cosh \rho \sinh t \partial_\phi \\ L_1 - L_{-1} &= 2 \sinh \rho \cosh t \partial_t + 2 \cosh \rho \cosh t \partial_\phi \end{aligned} \quad (177)$$

The other static patch is obtained from this one by sending $t \rightarrow t + \pi i$.

Inflating coordinates: These obey

$$\begin{aligned} X_0 &= \frac{1}{2}(-\eta^{-1} + \eta - x^2 \eta^{-1}) \\ X_1 &= \frac{1}{2}(\eta^{-1} + \eta - x^2 \eta^{-1}) \\ X_2 &= x \eta^{-1} \end{aligned} \quad (178)$$

with metric

$$ds^2 = \frac{1}{\eta^2} (-d\eta^2 + dx^2). \quad (179)$$

The $SL(2, R)$ action is induced from the other two via the coordinate transformations below. We only notice that

$$L_0 = -2\eta \partial_\eta - 2x \partial_x. \quad (180)$$

The three coordinate systems are related via the following light-cone maps

$$(\eta \pm x)^2 = e^{-2(t \pm \rho)} = \frac{1 + \sin(\tau \mp \phi)}{1 - \sin(\tau \mp \phi)} \quad (181)$$

which is what we would expect for a massless field in two dimensions. Since the static coordinates on the southern and northern diamonds are related via $t \rightarrow t + \pi i$, it is clear that a global mode which is continued from north to south with behavior $\exp(i\omega t)$ picks up a factor of $\exp(\pm \pi \omega)$, which shows the thermal nature of the vacuum defined with respect to these modes.

In global coordinates, the modes induced from the static patch are

$$\Phi_\omega^\pm = \left(\frac{1 + \sin(\tau \mp \phi)}{1 - \sin(\tau \mp \phi)} \right)^{2i\omega}. \quad (182)$$

These modes obey

$$\begin{aligned}
L_0 \Phi_\omega^\pm &= -8i\omega \Phi_\omega^\pm \\
L_1 \Phi_\omega^\pm &= \mp 4i\omega \Phi_{\omega-i/4}^\pm \\
L_{-1} \Phi_\omega^\pm &= \pm 4i\omega \Phi_{\omega+i/4}^\pm.
\end{aligned} \tag{183}$$

The modes (182) are indeed of the form given in (165), once we take $\tau = -\pi/2$, rescale ω , and identify y_1 with $\cos \phi$.

In the global patch the quantization is exactly as for the closed string. The conformal factor drops out of the action and also out of the KG inner product. Let us look for a state that is invariant under $SL(2, R)$. If L_k^l are the standard Virasoro generators for the left movers, and L_k^r the standard Virasoro generators for the right movers, then the $SL(2, R)$ we want to preserve is

$$L_1 = L_1^l + L_{-1}^r, \quad L_{-1} = L_{-1}^l + L_1^r, \quad L_0 = L_0^l - L_0^r \tag{184}$$

This is structure similar to what we see for a boundary state, and we can easily write down the relevant $SL(2, R)$ invariant state in some form $\exp(\sum_n a_{-n}^l a_{-n}^r)|0\rangle$. This illustrates the analogy between boundary states, and the $SO(d, 1)$ invariant vacuum states that we find in de Sitter space. Perhaps the one-parameter family of de Sitter invariant vacua is closely related to the one-parameter family of boundary states for a free scalar field predicted a long time ago by Friedan, see also recent paper [79].

Finally, we notice that if we compute the Klein-Gordon inner product between Euclidean modes and the L_0 eigenmodes in (182), we run into integrals precisely of the form (170). This is further evidence for the idea that the static patches are simply distinguished by the sign of ω , as we suggested above.

5 Conclusions

In this paper we have explored various physical and mathematical problems inspired by possible holographic descriptions of de Sitter space. First, in view of the similarities between de Sitter space and Euclidean anti-de Sitter space, we studied to what extent data on one can be mapped onto the other. We showed that there is a nonlocal map that commutes with the de Sitter isometries and transforms the bulk-boundary propagator and solutions of the free wave equation in de Sitter onto the same quantities in Euclidean anti-de Sitter. This map also transforms the two de Sitter boundaries into the single Euclidean AdS boundary via an antipodal identification as advocated in [4]. This raises the possibility that the holographic dual to EAdS could also describe de Sitter, but since the map has a nontrivial kernel, our study suggests instead that a holographic dual to de Sitter would involve independent (but possibly entangled) CFTs associated with both de Sitter boundaries. We reached a similar conclusion by

studying the action as a functional of boundary data for classical scalar fields in dS, as well as for 3d de Sitter gravity in both the Einstein and Chern-Simons formulations. As part of the exploration we displayed a family of solutions to 3d gravity with a positive cosmological constant in which the equal time sections are arbitrary genus Riemann surfaces.

In the second part of the paper we argued that if de Sitter space is dual to a Euclidean CFT, then the field theory would have novel hermiticity conditions, and realize a different form of conformal symmetry that we called the *Euclidean Virasoro algebra*. Since there is no concrete proposal for a field theory dual to de Sitter, we explored the Euclidean Virasoro symmetry in the context of a free boson and went a long way towards establishing properties of such theories. In addition to exploring unitary realizations of the symmetry, the structure of the spectrum and partition function, we discussed how we anticipate the novel CFTs to play a role in describing de Sitter space. For example, we showed that our new hermiticity conditions it is possible to construct an $SL(2, C)$ invariant state in a product of two $SL(2, C)$ invariant theories. Such a state is a candidate for a dual description of the de Sitter vacuum. We also argued it might be possible to rewrite the product of two such Hilbert spaces realizing Euclidean Virasoro symmetries, as a product of highest and lowest weight representations. Neither of the latter would realize the $SL(2, C)$ symmetry of de Sitter space unitarily but this split would be very natural for describing the physics in antipodal static patches of de Sitter space, with thermal entanglement between the highest/lowest weight factors giving rise to de Sitter thermodynamics. Finally we provided evidence for this picture by examining the physics of a scalar field in de Sitter space, explaining how the dS isometries are realized on it and how mode solutions in the global and static patches are related.

Our picture is also able to accomodate other asymptotically de Sitter spaces. For example, the spinning conical defects in 3d de Sitter [61, 62] would simply correspond to different entangled states. We would expect these states to have lower entropy than de Sitter when decomposed into representations appropriate to antipodal static patches. It is possible that thinking about the representation theory of such entangled states that preserve the asymptotic de Sitter symmetries would shed light on the de Sitter mass bound conjecture of [6]. If there is a dS/CFT correspondence, there is a relation between time evolution in spaces with a positive cosmological constant and RG flow in the dual field theory [77, 6, 78]. In our picture, any asymptotically dS space would correspond to an entangled state in a suitable product of CFTs, and we would compute a flow in this state to holographically describe time evolution. In general, if we compute the quantum gravity path integral between specified asymptotically de Sitter boundary conditions we would expect to sum over all intermediate geometries, including singular and topologically disconnected ones. Possibly these additional spaces have a relation to the mass bound conjecture of [6] and appear in the dual CFT spectrum above the de Sitter bound.

It has been suggested that there cannot be a dS/CFT correspondence at all [80]. The argument in [80] started with dS space with a cutoff in time, and with two operators, one on each boundary, and each part of the same static patch. As we take the boundaries to infinity, quantum gravity corrections will induce a finite tail in the two-point function, and therefore the asymptotic scaling behavior needed for a dS/CFT correspondence does not exist, according to [80].

This quantum gravity calculation is obviously hard to do. In [80], the finite tail is obtained by assuming that we are computing in a thermal ensemble with a discrete spectrum. Since the relevant quantum gravity calculations probe the full global structure of de Sitter space, we need to assume that global de Sitter is described by a thermal state in a theory with a discrete spectrum. In our setup, however, global de Sitter is described by a pure state, and the finite entropy is due to entanglement. Therefore, if we do the calculation suggested in [80], we will most likely not find a finite tail. There are complicated, non-local operators in the dual theory that do probe the finite tail and the thermal nature of the static patch. These are obtained by tracing over a suitable part of the Hilbert space, for which a preliminary proposal was given in section 4.3. It is these operators that describe the static patch, but they are not the conventional local operators in the boundary theory.

This work has an exploratory character and there are many loose ends and potential directions forward. For example, if CFTs exist with the novel hermiticity conditions that we have described, can they be used to define new string theories? We describe this possibility in greater detail below and then proceed to enumerate a number of worthwhile directions for future work.

5.1 New string theories?

Can there exist new string theories based on non-canonical reality conditions discussed above, ignoring for the moment the problems with the spectrum and partition function in the toy model discussed in Sec. 3? One possibility is that such string theories underlie cosmological spacetimes like de Sitter. In this regard it is interesting to investigate the hermiticity conditions in a theory where we T-dualize a time direction as proposed by Hull [1]. We leave the detailed study of this question for future work, but here we give an heuristic argument that new reality conditions can give imaginary B-fields as in Hull's theories.

First, we change the hermiticity from $L_n^\dagger = -\bar{L}_n$ to $L_n^\dagger = \bar{L}_{-n}$. This gives a theory with $SL(2, C)$ symmetry rather than $SL(2, R) \times SL(2, R)$. Notice that this is the usual hermiticity condition of string theory, except that right and left movers are interchanged. Normally, the graviton and B-field vertex operators are

$$\epsilon_{\mu\nu} \alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu |k\rangle \quad (185)$$

and this is a real field, because the complex conjugate (not the hermitian conjugate)

of α_{-1} is itself. The new hermiticity conditions are the same as the old ones, except that left and right movers are also interchanged in the process. It is reasonable that the same is true for complex conjugation. Hence, the vertex operator above goes under complex conjugation over into

$$\epsilon_{\nu\mu}^* \alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu |k\rangle. \quad (186)$$

In the usual case we had $\epsilon_{\mu\nu}^*$ as prefactor, and both the graviton and antisymmetric tensor have a real polarization. In the present case, the graviton polarization still is real, but the B-field polarization is imaginary, exactly as in Hull's theories which contain de Sitter space.

Even though this observation is hardly a proof that we need non-canonical hermiticity conditions in order to construct string theories of asymptotically de Sitter spaces (and perhaps more general time-dependent backgrounds), this idea warrants further investigation. Other recent interesting attempts to find de Sitter space in string theory involve non-critical string theories [18, 19] and spacelike branes [31].

5.2 Open Problems

Given the exploratory nature of this work it is appropriate to conclude with a list of open problems raised by our studies:

- Study the extension of the nonlocal map from dS to EAdS to the full interacting supergravity.
- Explore other realizations of the non-canonical hermiticity conditions to find examples in which the spectrum and partition function are fully under control.
- Explicitly show how unitary representations of the Euclidean Virasoro algebra based on the principal series for $SL(2, C)$ can be decomposed into products of highest and lowest weight Virasoro representations, as expected from de Sitter physics. Likewise study whether entangled states in a product of highest and lowest weight representations can lead to a 1-parameter family of $SL(2, C)$ -invariant states in a product of two unitary Euclidean Virasoro representations, or whether there is a unique $SL(2, C)$ -invariant entangled state corresponding to the Euclidean vacuum of de Sitter space.
- Explain de Sitter thermodynamics using the novel dual CFTs introduced here.
- Explore the extension of the structures discussed in this paper to D dimensions and to other time dependent backgrounds. In particular study whether and how time can be holographically generated in cosmological backgrounds via RG flows of a dual Euclidean theory [6, 77, 78].

- Study string theories based on CFTs with the new hermiticity conditions. Do they naturally arise from T-duality in time as in Hull’s work [1]? Can such string theories apply to more general time-dependent backgrounds?

Of course, the single most important open problem in this area is to find de Sitter space or any other expanding universe as a controlled solution to string theory. A concrete stringy realization of such spaces and their holographic duals would teach us a great deal about the role of time in quantum gravity.

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